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Isomer Energy System Design Concepts

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1. Introduction and Summary

Isomer energy supplies offer the potential to increase the output power over time to accommodate varying power needs. Other materials with similar energy density (for example isotopic energy sources such as ^{238}Pu) do not offer the potential to increase power with time. Often the design life of an energy source is significant when compared to the half-life of the isotope. As a result, the conventional isotopic energy supplies operate with significant excess power at start of life to meet the power needs at end-of-life. For example a ^{238}Pu radioisotope energy supply with a 35-year design life must account for radioactive decay losses of about 25% of the plutonium present at the start of life. This decay loss is significant if the required output power from the device is constant with time. If the required power output from the device increases with time (such as some space applications), significant increases in power supply weight are required to meet design power requirements.

Isomer energy supplies offer the potential to increase the output power over time to meet varying power needs, thereby offering a significant advantage over conventional systems. Isomer energy supplies also offer the possibility of being "turned on" based on need at a specific time. These characteristics offer distinct advantages to isomer energy supplies.

This report examines the basic engineering characteristics of a hypothetical isomer energy supply in order to gain insight into properties of isomers that will make them potentially useful as energy sources in engineered systems. These isomer properties provide a basis for identification of candidate isomers and provide a basis for an isomer search.

Two parameters that include both nuclear properties of the isomer and properties of an engineered system are key to system performance. The first parameter is the energy gain of the isomer energy system. The energy gain for an isomer system is defined as the ratio of

$$\frac{\text{Useful energy out}}{\text{Useful energy in}}$$

The energy gain, β, is given by the dimensionless parameter:

$$\beta = \frac{\eta_{trigger}\ \eta_{out}\ B\ E_{decay}}{E_{trigger}} \tag{1}$$

Where

$\eta_{trigger}$ is the efficiency of the trigger or excitation mechanism (dimensionless)

η_{out} is the efficiency of a thermal to electric conversion system producing electricity from the energy supply (dimensionless)

B is the branching ratio for the isomer decay mode of interest (dimensionless)

E_{decay} is the total released energy, per nucleus or isomer, from decay of the isomer (eV per atom)

$E_{trigger}$ is the total energy required to populate the excited isomeric state from, e.g., the ground state or alternatively, the energy to trigger instantaneous isomer decay (eV per decay)

The parameters identified in equation 1 are shown below.

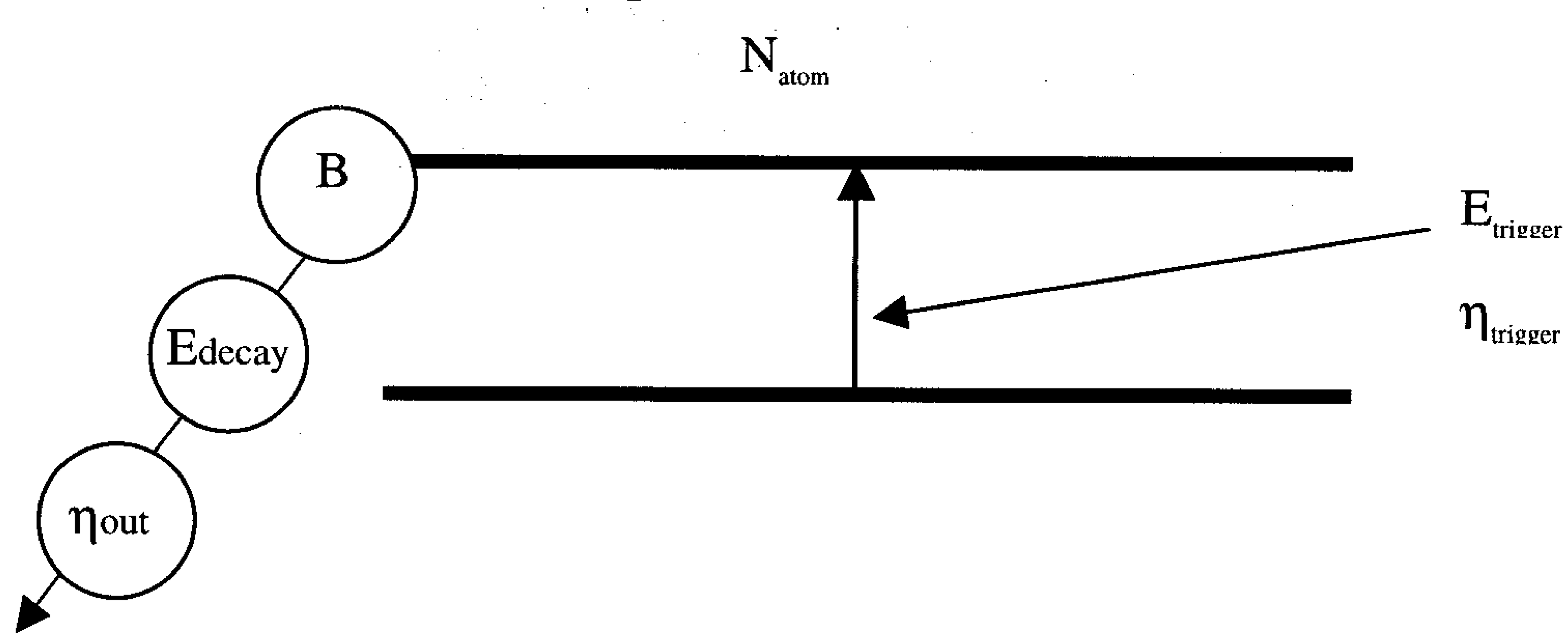


Figure 1. System under consideration.

The characteristic time for many practical systems is related to $1/(\beta-1)\,\lambda$, where β is the energy gain and λ is the decay constant for the isomer state. The minimum time to double the instantaneous power output, t_{double}, for the isomer energy system can be derived as :

$$t_{double} = \frac{\ln(2)}{(\beta-1)\,\lambda} \tag{2}$$

Note that for conceptual systems where the input energy triggers nuclear isomer decay (such as "superradiance"), the parameter λ is very large, indicating that such a conceptual isomer energy system will allow a very rapid power increase.

The excited isomer state can be selectively triggered to control system power output. If a fraction of atoms are excited initially, constant or increasing power could be produced over a very long time by using a small portion of the output energy to trigger additional atoms. This feedback method would allow:

- Constant power output for a long period of time. In many applications a constant power output is desirable. Conventional isotopic power supplies experience exponential decay in power output.
- Power that increases proportional to t^2. In some space applications, it is desirable to deliver a constant strength signal to earth from an outbound satellite. An energy supply with power output that increases in time, as t^2, would have significant value in such an application. In concept, an isomer energy system could have this capability.
- Output power that cycles on and off with a pre-determined power shape appropriate for a given application. In concept, an isomer power supply can deliver tailored output power, e.g., in the form of a step function, square wave or a saw tooth function.

The search for an appropriate isomer state has not yet identified an isomer with desired properties. But if an appropriate isomer is identified, reasonable quantities of that isomer could be very practical energy sources.

The remainder of this report provides a summary of the derivation of the relationships given above and provides examples of several conceptual isomer energy systems. Chapter 2 derives the expressions for energy gain and minimum power output doubling time for an isomer energy system. Chapter 3 provides a concept for constant power output. Chapter 4 provides a concept for an energy supply with power output that increases as the square of time. Chapter 5 derives a relationship for the general case where power output varies as a function of time.

2. Energy Gain and Power Doubling Time

A conceptual isomer system could produce energy to power a local device. One possible energy output is electrical. Key parameters related to energy flow for a conceptual energy system are provided in Figure 1.

The energy balance for this conceptual device includes the following components:

- Nuclear energy is stored in the isomer. If N_{atom} are in an excited state that releases energy of E_{decay} per decay with a branching ratio of B, the stored nuclear energy (that will ultimately be released on decay) is B E_{decay} N_{atom}. Normally the nuclear decay energy is converted to thermal energy, which is converted to electrical energy to power a device. If the nuclear decay to electrical energy conversion efficiency of this device is η_{out}, the electrical energy available from excited atoms is:

$$E_{store} = \eta_{out}\, B\, E_{decay}\, N_{atom} \qquad (3)$$

  To be consistent with the units of η_{out}, B, E_{decay} and N_{atom}, the term E_{store} is normally expressed in units of eV for the system. With an appropriate conversion factor E_{store} can be expressed in MKS units of joules (1.60207 E-19 joules per eV).

- Energy may be required to produce an excited nuclear state or trigger energy release from an excited state. The energy to excite or trigger N_{atom} of an isomer is

$$E_{in} = (E_{trigger}/\eta_{trigger})\, N_{atom} \qquad (4)$$

  where $E_{trigger}$ is the <u>nuclear energy</u> required to excite an energy release, and $\eta_{trigger}$ is the efficiency of converting input energy into nuclear energy. Note that under some trigger mechanisms, the energy input occurs over an extended time. To be consistent with the units of $E_{trigger}$, $\eta_{trigger}$ and N_{atom}, the term E_{in} is in units of eV. With an appropriate conversion factor E_{in} can be expressed in units of joules.

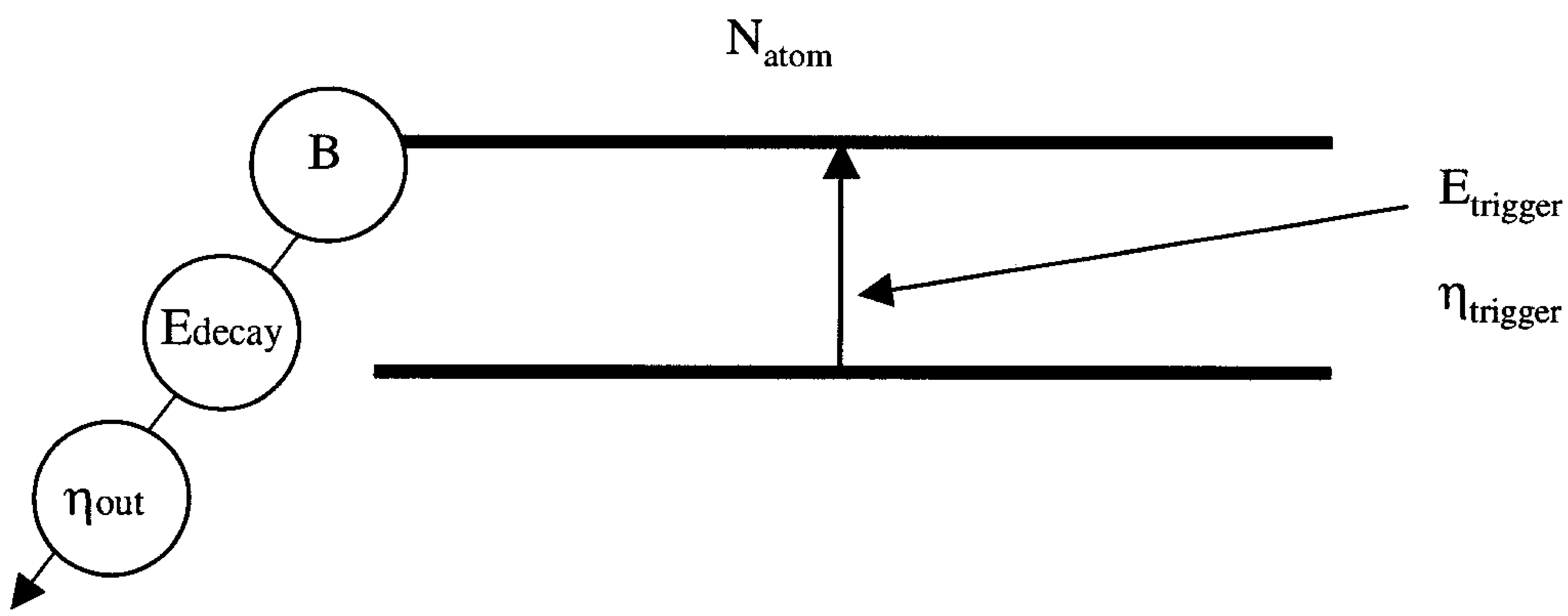


Figure 1. System under consideration.

The energy gain of the conceptual system is defined as the energy input to the system divided by the energy that ultimately is produced. In the example cited the energy gain, β, is given by:

$$\text{Energy Gain} = \beta = E_{store} / E_{in} \tag{5}$$

$$\beta = \eta_{out}\, B\, E_{decay}\, N_{atom} / [(E_{trigger}/\eta_{trigger})\, N_{atom}] \tag{6}$$

$$\beta = \eta_{trigger}\, \eta_{out}\, B\, E_{decay} / E_{trigger} \tag{7}$$

The energy gain is a key parameter for conceptual isomer power systems. Practical isomer power systems require an energy gain, β, greater than 1.

The energy gain is one factor that determines how quickly the power can be increased in a conceptual isomer system. In some applications, it may be desirable to "turn on" an isomer energy supply and increase output power. This can be done be using a feedback loop to return energy from the energy supply output to the energy input to excite or trigger additional isomer decays. For the conceptual isomer system shown schematically in Figure 2, assume that at time 0, an initial energy input, $E_{in}(0)$, (delivered in a short time period) turns on the isomer energy system by exciting some isomer atoms. The limiting rate of power output increase can be determined for this conceptual system if one assumes that all of the output power is fed back into the device to excite or trigger additional isomer atoms.

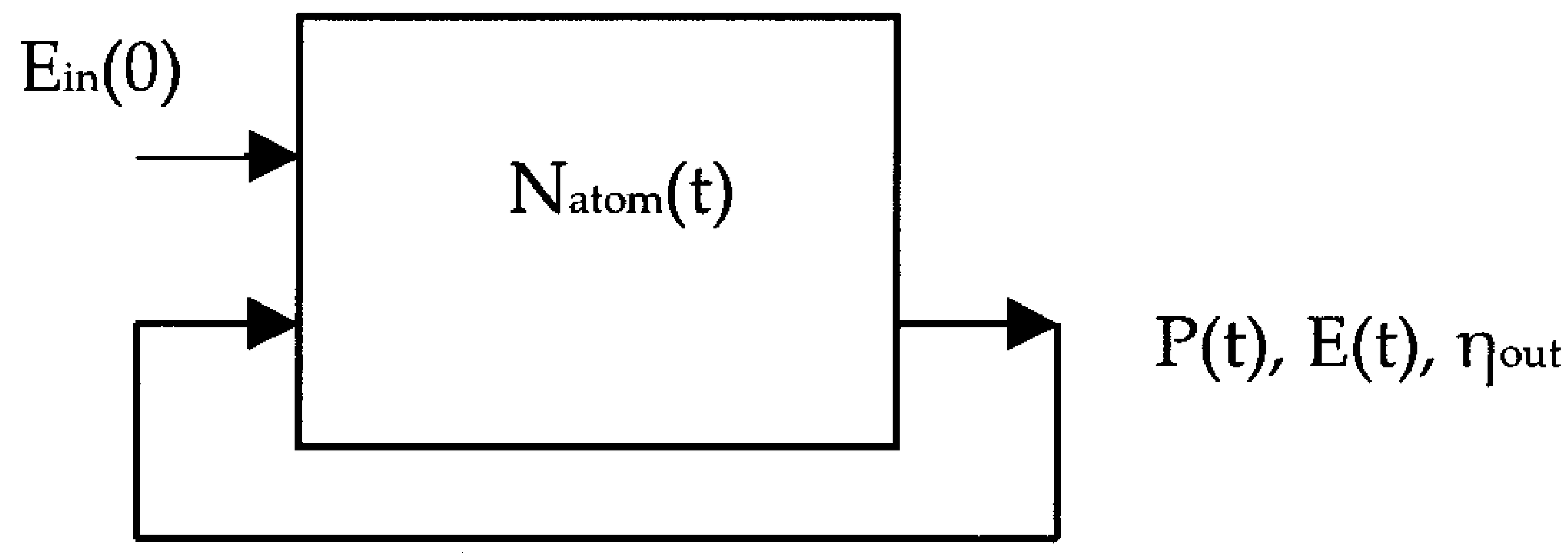


Figure 2. A conceptual isomer energy supply system with a feedback loop

In this system, an initial energy release, $E_{in}(0)$, results in initial excitation of isomer atoms (see equation 4)

$$N_{atom}(0) = E_{in}(0)\ \eta_{trigger} / E_{trigger} \tag{8}$$

The decay of the excited atoms results in power output given by

$$P(t) = \lambda\ \eta_{out}\ B\ E_{decay}\ N_{atom}(t) \tag{9}$$

In equation 9, the units of power, P(t) are eV per second (assuming the decay constant, λ, is expressed in inverse seconds). The expression for power can be integrated over time to provide the energy output

$$E(t) = \int_0^t \lambda\ \eta_{out}\ B\ E_{decay}\ N_{atom}(t)\ dt \tag{10}$$

If the entire energy output, E(t), is used to excite or trigger additional isomers, the number of atoms that are excited by this "feedback energy" is

$$N_{atom}(t) = \eta_{trigger} / E_{trigger}\ E(t) \tag{11}$$

The number of excited atoms at time t is given by subtracting the number of decayed atoms from the number that have been excited

$$N_{atom}(t) = \eta_{trigger} / E_{trigger}\ E(t) - \int_0^t \lambda\ N_{atom}(t)\ dt \tag{12}$$

Substituting equation (10) into equation (12),

$$N_{atom}(t) = \eta_{trigger} / E_{trigger} \int_0^t \lambda\, \eta_{out}\, B\, E_{decay}\, N_{atom}(t)\, dt - \int_0^t \lambda\, N_{atom}(t)\, dt \qquad (13)$$

Which can be further simplified by substituting for the expression β from equation (1),

$$N_{atom}(t) = \int_0^t (\beta-1)\, \lambda\, N_{atom}(t)\, dt \qquad (14)$$

This expression can be solved (for example by using Laplace transforms) for the initial condition provided in equation (8) to yield

$$N_{atom}(t) = N_{atom}(0)\ e^{(\beta-1)\lambda t} \qquad (15)$$

The instantaneous power output of this system can be determined from equations (9) and (15)

$$P(t) = \lambda\, \eta_{out}\, B\, E_{decay}\, N_{atom}(0)\ e^{(\beta-1)\lambda t} \qquad (16)$$

$$P(t) = P(0)\, e^{(\beta-1)\lambda t} \qquad (17)$$

From equation (10), it can be shown that the energy output is equal to

$$E(t) = \int_0^t \lambda\, \eta_{out}\, B\, E_{decay}\, N_{atom}(0)\ e^{(\beta-1)\lambda t}\ dt \qquad (18)$$

Integrating equation (18) to simplify

$$E(t) = \frac{\lambda\, \eta_{out}\, B\, E_{decay}\, N_{atom}(0)}{(\beta-1)\,\lambda}\ e^{(\beta-1)\lambda t} \qquad (19)$$

Substituting equation (8) into equation (19)

$$E(t) = \frac{\lambda\, \eta_{out}\, \eta_{trigger}\, B\, E_{decay}\, E(0)}{(\beta-1)\,\lambda\, E_{trigger}}\ e^{(\beta-1)\lambda t} \qquad (20)$$

Substituting for β and simplifying

$$E(t) = \frac{\beta\, E(0)}{(\beta-1)}\ e^{(\beta-1)\lambda t} \qquad (21)$$

In the preceding equations, the expression $(\beta\text{-}1)\ \lambda$ is related to the maximum rate of increase of energy output, power output and number of isomer. The minimum time to double energy output, power output or number of atoms in the excited state in a conceptual isomer energy system is

$$t_{double} = \ln(2) / ((\beta - 1)\ \lambda) \tag{22}$$

Therefore, large values of $(\beta\text{-}1)\ \lambda$ will provide the possibility of increasing power output more quickly in a conceptual isomer energy system. By inspection of equation (22), large values of energy gain, β, and decay constant, λ, yield shorter doubling times and more operating flexibility. Note that as λ approaches infinity, the product, $(\beta\text{-}1)\ \lambda$, also becomes infinite.

3. Steady State Constant Power Output

In many practical energy supply systems, it may desirable to have a constant power output. Unlike a conventional radioisotope energy source, a conceptual isomer power supply system has the potential to provide a constant power output until the inventory of isomer is depleted.

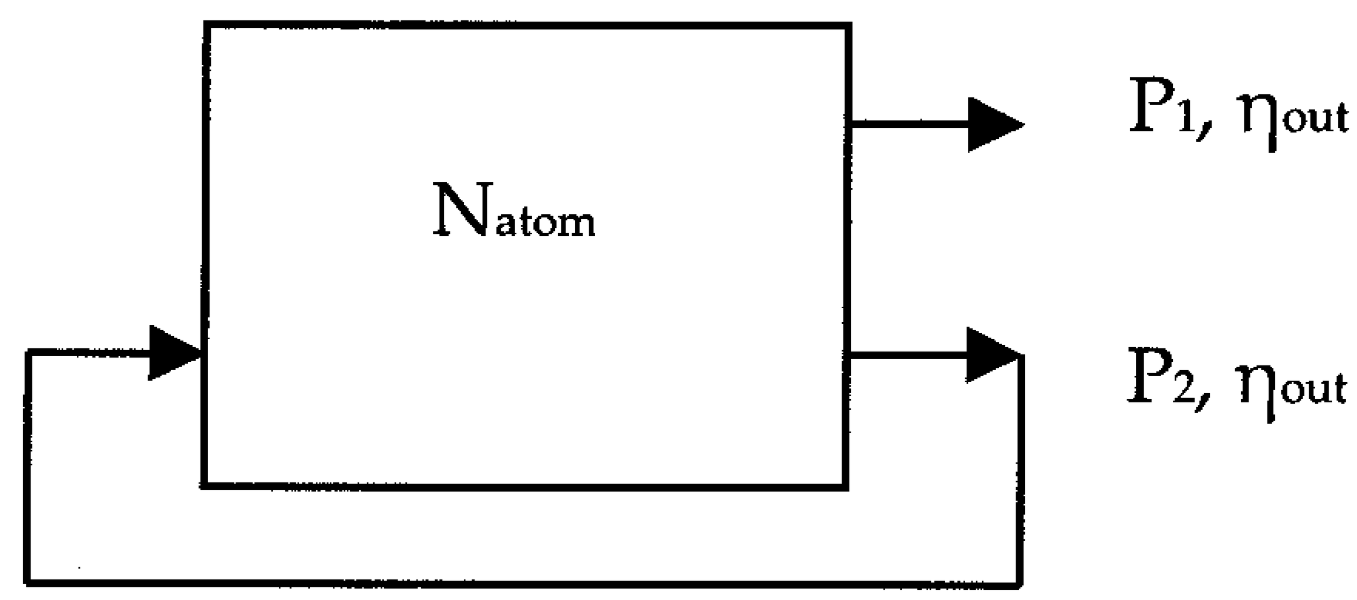


Figure 3. A conceptual isomer energy supply with energy feedback loop

A conceptual power supply is illustrated in Figure 3. In this system, there are two power outputs. One power output, P_1, is available to supply energy to a local device. The second power output, P_2, is used to feedback into the isomer power supply to excite or trigger additional isomer atoms. The total power output of the device is given by

$$P_{out} = P_1 + P_2 \tag{23}$$

From equation (9), we find that

$$P_{out} = \lambda\, \eta_{out}\, B\, E_{decay}\, N_{atom} \tag{24}$$

We know that from decay of the excited isomer,

$$dN_{atom}/dt = -\lambda\, N_{atom} \tag{25}$$

Therefore, the power output P_2, must be sufficient to trigger or excite atoms at the rate in equation (25). Differentiating equation (11), and substituting equation (25) for dN_{atom}/dt we find that:

$$P_2 = \lambda\, N_{atom}\, E_{trigger}/\eta_{trigger} \tag{26}$$

Substituting equations (26) and (24) into equation (23),

$$P_1 + \lambda N_{atom} E_{trigger}/\eta_{trigger} = \lambda \eta_{out} B E_{decay} N_{atom} \tag{27}$$

Solving for N_{atom},

$$N_{atom} = \frac{P_1}{\lambda (\eta_{out} B E_{decay} - E_{trigger}/\eta_{trigger})} \tag{28}$$

Substituting an expression for N_{atom} derived from equation (26) into equation (28),

$$P_2 = \frac{\lambda P_1 E_{trigger}/\eta_{trigger}}{\lambda (\eta_{out} B E_{decay} - E_{trigger}/\eta_{trigger})} \tag{29}$$

Simplifying, substituting for β from equation (1) and solving for P_2,

$$P_2 = P_1 / (\beta - 1) \tag{30}$$

Since the energy gain, β, is large for practical systems, only a small feedback power, P_2, is required to maintain constant power output for this conceptual system.

4. Power Output Increasing as t^2

For the idealized case of a satellite traveling outbound from earth at constant velocity, v, the distance, s, traveled at time t is

$$s = v\,t \tag{31}$$

The satellite power needed to transmit an equivalent strength signal in watts/m^2 to an earth station will increase as the square of distance, s^2, from the earth. If the satellite is traveling at constant velocity, then from equation (31), the signal must increase as the square of time, t^2, to provide constant signal strength to earth. If a signal broadcast power, P_o, results in adequate signal strength at time t_o, then at time $t > t_o$, the power required for an equivalent signal on earth at a later time, t, is:

$$P_1(t) = P_o\, t^2 / t_o^2 \tag{32}$$

Letting

$$\alpha = P_o / t_o^2 \tag{33}$$

Equation (32) becomes,

$$P_1(t) = \alpha\, t^2 \tag{34}$$

Figure 3 provides a schematic view of a system concept to produce an output power that will increase as t^2 for a broadcast signal from an outbound satellite.

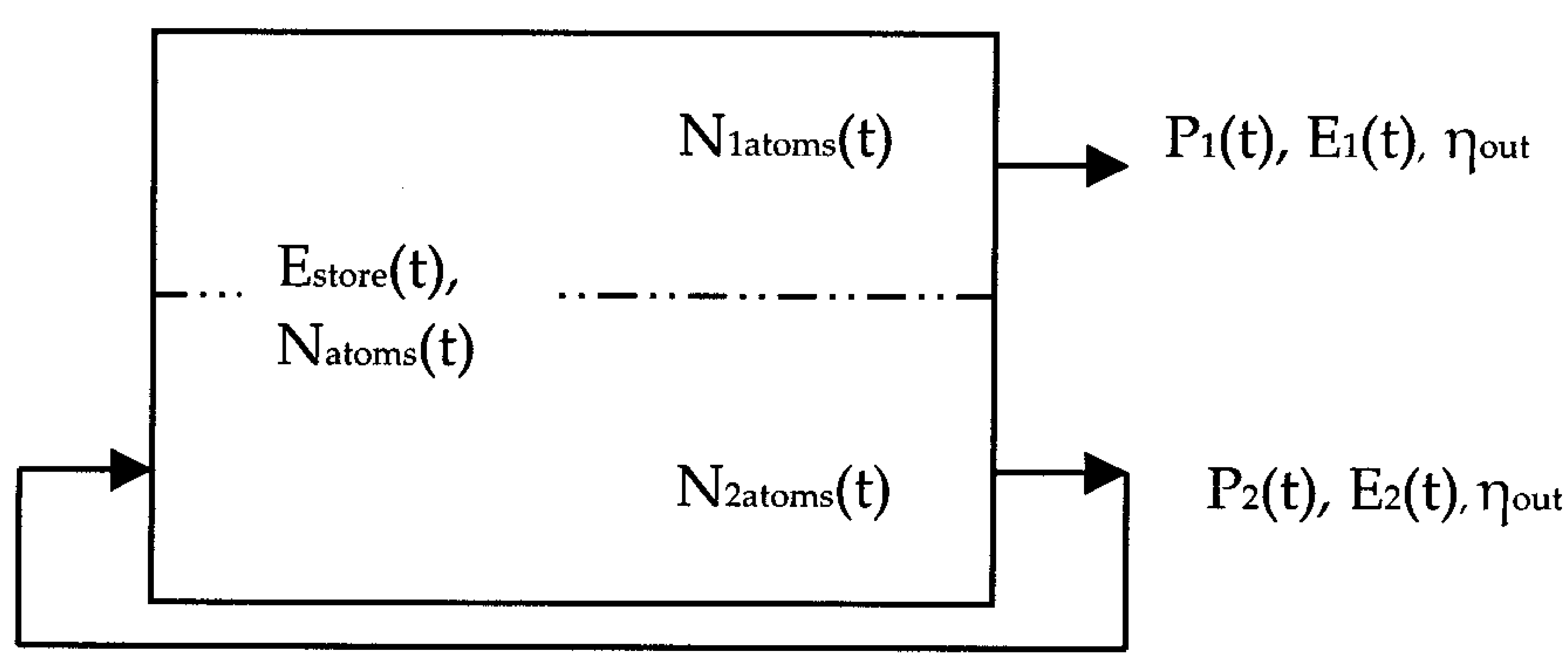


Figure 3 Conceptual isomer energy supply with energy storage indicated

In this schematic,

$N_{1atom}(t)$ is the number of triggered or excited state atoms that are required to provide the power output $P_1(t)$
$N_{2atom}(t)$ is the number of triggered or excited state atoms that are required to provide the power output $P_2(t)$
$N_{atom}(t) = N_{1atom}(t) + N_{2atom}(t)$ is the total number of triggered or excited state atoms
$P_1(t)$ is the power output used to power a local device. In this example, $P_1(t)$ increased at a rate proportional to t^2 (see equation 34).
$P_2(t)$ is the power output used for feedback into the isomer power system. The energy from P_2 results in additional excited atoms.
$P_{out}(t) = P_1(t) + P_2(t)$
$E_1(t)$ is the energy output used to power a local device.
$E_2(t)$ is the energy output used for feedback into the isomer power system.
$E_{out}(t) = E_1(t) + E_2(t)$
$E_{store}(t)$ is the stored potential nuclear energy in the triggered or excited isomer states.

An energy balance can be performed on the system. Noting that the feedback energy, $E_2(t)$, result in isomer potential energy of $\beta\ E_2(t)$, an energy balance on the power system yields,

$$\beta\ E_2(t) = E_{store}(t) + E_{out}(t) \tag{35}$$
$$\beta\ E_2(t) = E_{store}(t) + E_1(t) + E_2(t) \tag{36}$$

Rearranging terms in equation (36),

$$(\beta - 1)\ E_2(t) = E_{store}(t) + E_1(t) \tag{37}$$

Expressions for $P_1(t)$, $N_{1atom}(t)$ and $E_1(t)$ can be derived. Combining equations (9) and (34) and simplifying,

$$P_1(t) = \lambda\ \eta_{out}\ B\ E_{decay}\ N_{1atom}(t) = \alpha\ t^2 \tag{38}$$

$$N_{1atom}(t) = \frac{\alpha\ t^2}{\lambda\ \eta_{out}\ B\ E_{decay}} \tag{39}$$

$$N_{1atom}(t) = \frac{(\eta_{trigger} / E_{trigger})\, \alpha\, t^2}{\lambda\, \beta} \qquad (40)$$

Equation (34) can be integrated (from 0 to time t, noting that $E_1(0) = 0$) to provide an expression for E_1,

$$E_1(t) = \int_0^t P_1(t)\, dt = \int \alpha\, t^2\, dt = \alpha\, t^3/3 \qquad (41)$$

An expression for $E_{store}(t)$ can be derived from equation (3)

$$E_{store}(t) = \eta_{out}\, B\, E_{decay}\, [N_{1atom}(t) + N_{2atom}(t)] \qquad (42)$$

The instantaneous power, $P_2(t)$ can be determined from equation (9)

$$P_2(t) = \lambda\, \eta_{out}\, B\, E_{decay}\, N_{2atom}(t) \qquad (44)$$

The output energy, $E_2(t)$, can be determined by integrating equation (44)

$$E_2(t) = \int_0^t P_2(t)\, dt = \lambda\, \eta_{out}\, B\, E_{decay} \int_0^t N_{2atom}(t)\, dt \qquad (45)$$

The output energy, $E_2(t)$, is fed back into the conceptual isomer energy system to trigger or excite additional isomer atoms. Equation (46) below is an atom balance on excited isomers. It is based on equations (25) and the derivative of equation (11) and describes the decay of isomer atoms and the production of new isomer atoms by the input energy, E_2.

$$d\, [N_{1atom}(t)+N_{2atom}(t)] / dt = -\lambda\, [N_{1atom}(t)+N_{2atom}(t)] + E_{trigger}/\eta_{trigger}\, dE_2(t) / dt \qquad (46)$$

Substituting equation (44) for $dE_2(t) / dt$ and grouping terms for β (equation 1),

$$d\, [N_{1atom}(t)+N_{2atom}(t)] / dt = -\lambda\, [N_{1atom}(t)+N_{2atom}(t)] + \beta\, \lambda\, N_{2atom}(t) \qquad (47)$$

$$d\, N_{2atom}(t)/dt - (\beta - 1)\, \lambda\, N_{2atom}(t) = -d\, N_{1atom}(t)/dt - \lambda\, N_{1atom}(t) \qquad (48)$$

Substituting equation (40) for $N_{1atom}(t)$, differentiating and regrouping terms,

$$d\, N_{2atom}(t)/dt - (\beta - 1)\, \lambda\, N_{2atom}(t) = -\frac{(\eta_{trigger} / E_{trigger})\, \alpha\, (\lambda\, t^2 + 2\, t)}{\lambda\, \beta} \qquad (49)$$

Equation (49) can be solved for $N_{2atom}(t)$,

$$N_{2atom}(t) = \frac{\alpha\, \eta_{trigger}\, ((\beta - 1)^2 \lambda^2\; t^2 + 2\, t\, \beta\, (\beta - 1)\, \lambda + 2\, \beta)}{E_{trigger}\, \beta\, (\beta - 1)^3\, \lambda^3} + C_o\, e^{(\beta-1)\lambda t} \tag{50}$$

The constant of integration, C_o, can be determined by solving equation (50) for t = 0 and noting that required for normal operations the minimum excess stored energy, E_{store}, will occur when the value of $C_o \approx 0$

$$C_o \quad = 0 \tag{51}$$

Substituting equations (50) and (51) into equation (44) gives

$$P_2(t) = \frac{\alpha\, \lambda\, \eta_{out}\, B\, E_{decay}\, \eta_{trigger}\, ((\beta - 1)^2 \lambda^2\; t^2 + 2\, t\, \beta\, (\beta - 1)\, \lambda + 2\, \beta)}{E_{trigger}\, \beta\, (\beta - 1)^3\, \lambda^3} \tag{52}$$

Substituting equation (1) for β into equation (52) and simplifying,

$$P_2(t) \quad = \frac{\alpha\, ((\beta - 1)^2 \lambda^2\; t^2 + 2\, t\, \beta\, (\beta - 1)\, \lambda + 2\, \beta)}{(\beta - 1)^3\, \lambda^3} \tag{53}$$

Equations (45) and (53) can be used to determine $E_2(t)$

$$E_2(t) = \frac{\alpha\, ((\beta - 1)^2 \lambda^2\; t^3/3 + t^2\, \beta\, (\beta - 1)\, \lambda + 2\, \beta\, t)}{(\beta - 1)^3\, \lambda^3} \tag{54}$$

The stored energy, $E_{store}(t)$, can be determined from equation (37) by substituting equation (41) for $E_1(t)$ and equation (54) for $E_2(t)$.

$$(\beta - 1)\;\; E_2(t) \;\; = E_{store}(t) \;\; + E_1(t) \tag{37}$$

$$E_{store}(t) \quad = \frac{\alpha\, ((\beta - 1)^2 \lambda^2\; t^3/3 + t^2\, \beta\, (\beta - 1)\, \lambda + 2\, \beta\, t)}{(\beta - 1)^2\, \lambda^3} \; - \; \alpha\;\; (t^3 / 3) \tag{55}$$

5. General case - power output F(t)

By inspection of the previous case (see equations 38, 40, 41), it can be shown that a generalized power output, $P_1(t) = F(t)$, is described by the following system of equations,

$$P_1(t) = F(t) \tag{56}$$

$$N_{1atom}(t) = \frac{(\eta_{trigger} / E_{trigger})\, F(t)}{\lambda\, \beta} \tag{57}$$

$$E_1(t) = \int_0^t P_1(t)\, dt = \int_0^t F(t)\, dt \tag{58}$$

By inspection of the previous case (see equation 48), $N_{2atom}(t)$ is the solution to the following equation,

$$dN_{2atom}(t)/dt - \lambda\ (\beta - 1)\ N_{2atom}(t) = -\frac{(\eta_{trigger} / E_{trigger})\ [\lambda\, F(t) + F'(t)]}{\lambda\, \beta} \tag{59}$$

Equation (59) can be solved for $N_{2atom}(t)$. The value of $N_{2atom}(t)$ can be used to calculate $P_2(t)$ and $E_2(t)$ using equations (44), (45) and (47)

$$P_2(t) = \lambda\, \eta_{out}\, B\, E_{decay}\, N_{2atom}(t) \tag{60}$$

$$E_2(t) = \lambda\, \eta_{out}\, B\, E_{decay} \int_0^t N_{2atom}(t)\, dt \tag{61}$$

$$(\beta - 1)\ E_2(t) = E_{store}(t) + E_1(t) \tag{62}$$

For many values of F(t, Equation (59) can be solved directly. Appendix A includes a tabulation of the solutions to the equations in this sections for $N_{1atom}(t)$, $E_1(t)$, $N_{2atom}(t)$, $P_2(t)$, $E_2(t)$ and E_{store} for several functional power outputs, $P_1(t)$, that are of potential interest for a hypothetical isomer energy supply. The results presented in Appendix A are for continuous functions.

Note that if F(t) is a periodic function, the values of individual functions $N_{1atom}(t)$ or $N_{2atom}(t)$ may be less than zero. This could result, for example, if N_{2atom} were used to generate E_1 output energy instead of E_2 output energy. However, in a physical system, the total number of isomer atoms -- $N_{atom}(t) = N_{1atom}(t) + N_{2atom}(t)$ -- cannot be less than 0. Based on analyses presented earlier, this conceptual system is subject to additional constraints on the rate of isomer atom increase and decrease. Due to the isomer decay constant, the maximum rate of excited isomer atom decrease is $-\lambda\, N_{atom}(t)$. The analyses presented in chapter 2 indicate that the maximum rate of isomer atom increase (without an external power supply) is $(\beta\text{-}1)\,\lambda\, N_{atom}(t)$. The equations below identify these constraints.

$$N_{atom}(t) > 0 \tag{63}$$

$$-\lambda \leq \frac{1}{[N_{atom}(t)]}\,\frac{d[N_{atom}(t)]}{dt} \leq (\beta\text{-}1)\,\lambda \tag{64}$$

Note that if the value of λ is very large (corresponding to a very short half-life for the isomer state), these limitations are may be insignificant for many engineering applications.

Equation (59) can also be solved by Laplace Transforms. Taking the Laplace transform

$$s\, n(s) - N_{2atom}(0) - (\beta - 1)\,\lambda\, n(s) = -\, \frac{(\eta_{trigger})\,[\lambda\, f(s) + s\, f(s) - F(0)]}{\lambda\, \beta\, E_{trigger}} \tag{65}$$

Where $n(s)$ is the Laplace Transform of $N_{2atom}(t)$ and $f(s)$ is the Laplace Transform of F(t). (Note that in equation (65), it is assumed that $F'(t) \neq 0$.) Equation (65) can be solved for $n(s)$.

$$n(s)\,(s - ((\beta - 1)\,\lambda)) = N_{2atom}(0) - \frac{(\eta_{trigger})\,[(s + \lambda)\, f(s) - F(0)]}{\lambda\, \beta\, E_{trigger}} \tag{66}$$

$$n(s) = \frac{N_{2atom}(0)}{(s - ((\beta - 1)\,\lambda))} - \frac{(\eta_{trigger})}{\lambda\, \beta\, E_{trigger}} \bullet \frac{[(s + \lambda)\, f(s) - F(0)]}{(s - ((\beta - 1)\,\lambda))} \tag{67}$$

In order to facilitate the Inverse Laplace Transform, the last term in equation (67) is factored using the method of partial fractions.

$$n(s) = \frac{N_{2atom}(0)}{(s - ((\beta - 1)\,\lambda))} - \frac{(\eta_{trigger})}{\lambda\, \beta\, E_{trigger}} \bullet \frac{[(s + \lambda - \beta\lambda + \beta\lambda)\, f(s) - F(0)]}{(s - ((\beta - 1)\,\lambda))} \tag{68}$$

The terms in equation (68) can be re-grouped and then simplified

$$n(s) = \frac{N_{2atom}(0)}{(s-((\beta-1)\lambda))} - \frac{(\eta_{trigger})}{\lambda\,\beta\,E_{trigger}} \bullet \frac{[(s-(\beta-1)\lambda)\ f(s)]}{(s-((\beta-1)\lambda))}$$

$$- \frac{(\eta_{trigger})\ (\beta\lambda)\ f(s)}{\lambda\,\beta\,E_{trigger}\ (s-((\beta-1)\lambda))} + \frac{\eta_{trigger}\,F(0)}{\lambda\,\beta\,E_{trigger}\,(s-((\beta-1)\lambda))} \qquad (69)$$

$$n(s) = \frac{N_{2atom}(0)}{(s-((\beta-1)\lambda))} - \frac{\eta_{trigger}}{\lambda\,\beta\,E_{trigger}}\ f(s) - \frac{(\eta_{trigger})\ (\beta\,\lambda\,f(s))}{\lambda\,\beta\,E_{trigger}\ (s-((\beta-1)\lambda))}$$

$$+ \frac{\eta_{trigger}\,F(0))}{\lambda\,\beta\,E_{trigger}\,(s-((\beta-1)\lambda))} \qquad (70)$$

Taking the inverse Laplace Transform and noting that $N_{2atom}(t)$ is the inverse Laplace Transform of n(s) in equation (70).

$$N_{2atom}(t) = N_{2atom}(0)\,e^{(\beta-1)\lambda t} - \frac{\eta_{trigger}}{\lambda\,\beta\,E_{trigger}}\ F(t)$$

$$- \frac{(\eta_{trigger})}{E_{trigger}}\ \mathcal{L}^{-1}\ \frac{f(s)}{(s-((\beta-1)\lambda))} + \frac{\eta_{trigger}}{\lambda\,\beta\,E_{trigger}}\ F(0)\,C_o\,e^{(\beta-1)\lambda t} \qquad (71)$$

where the symbol $\mathcal{L}^{-1}$ in equation (71) is the Inverse Laplace Transform of the function that follows. For many cases, the Inverse Laplace Transform of the third term in Equation (71) can be determined by reference to tabulations of Laplace Transforms. Equation (71) also can be evaluated by applying the Convolution (Faltung) Theorem for Laplace Transforms to evaluate the third term. Re-arranging terms and applying the Convolution Theorum yields

$$N_{2atom}(t) = N_{2atom}(0)\,e^{(\beta-1)\lambda t} - \frac{\eta_{trigger}}{\lambda\,\beta\,E_{trigger}}\ [F(t) - F(0)\,C_o\,e^{(\beta-1)\lambda t}\ ]$$

$$- \frac{(\eta_{trigger})}{E_{trigger}} \int_0^t e^{(\beta-1)\lambda(t-\tau)} F(\tau)\, d\tau \qquad (72)$$

For some applications, it may be useful to have a periodic power output, $P_1(t)$. Two cases are presented here. In the first case, the power output is the product of a function (for example $f(t) = t^2$) and a "on/off switch that turns on at time t_a and off at time t_b from the start of a cycle. The on/off cycle for this periodic function repeats at a regular time interval of t_c. Figure 4 below provides an example of this function that is referred to as a Type 1 function in this text. In the second case, the power output is a constant power shape that repeats at an interval of t_c. Figure 5 below provides and example of this function that is referred to as a Type 2 function in this text.

In equation (71), the final term can be evaluated as an indefinite integral to give

$$\int e^{(\beta-1)\lambda(t-\tau)} F(\tau)\, d\tau = G(t) \qquad (73)$$

The definite integral can be evaluated for Type 1 and 2 periodic functions noting that $F(t) = 0$ if $t < t_a$ and $F(t) = 0$ if $t_b < t < t_c$:

$$\int_0^{t_{max}} e^{(\beta-1)\lambda(t-\tau)} F(\tau)\, d\tau = \sum_{n=1}^{n_{cyclemax}} \int e^{(\beta-1)\lambda(t-\tau)} F(\tau)\, d\tau$$

$$= \sum_{n=1}^{n_{cyclemax}} [G(t_b + n\, t_c - 1) - G(t_a + n\, t_{cycle} - 1)] \qquad (74)$$

Where $n_{cyclemax} = t_{max} / t_c$

tmax = Energy source lifetime

The definite integral can be simplified for Type 2 and periodic functions by noting that the shape of the required output power is the same during each interval:

$$\int_0^{t_{max}} e^{(\beta-1)\lambda(t-\tau)} F(\tau)\, d\tau = \sum_{n=1}^{n_{cyclemax}} [G(t = t_b + n\, t_{cycle} - 1, t_b) - G(t = t_a + n\, t_{cycle} - 1, t_a)] \qquad (75)$$

Appendix B presents values of G(t) and $[G(t, t_b) - G(t, t_a)]$for several periodic functions of interest. The expressions in Appendix B can be used to evaluate Equation (71) for periodic functions of interest. Note that the sums of series represented by equations (73) and (74) can be simplified for many cases of interest.

For the cases not covered in the appendix, Equations (59), (70), or (71) can be solved to determine the expression for $N_{2atom}(t)$. The remaining parameters describing the system ($N_{1atom}(t)$, $E_1(t)$, $P_2(t)$, $E_2(t)$ and $E_{store}(t)$) can be determined from Equations (56), (57), (58), (60), (61), and (62).

Figure 4 Example of a type 1 periodic function

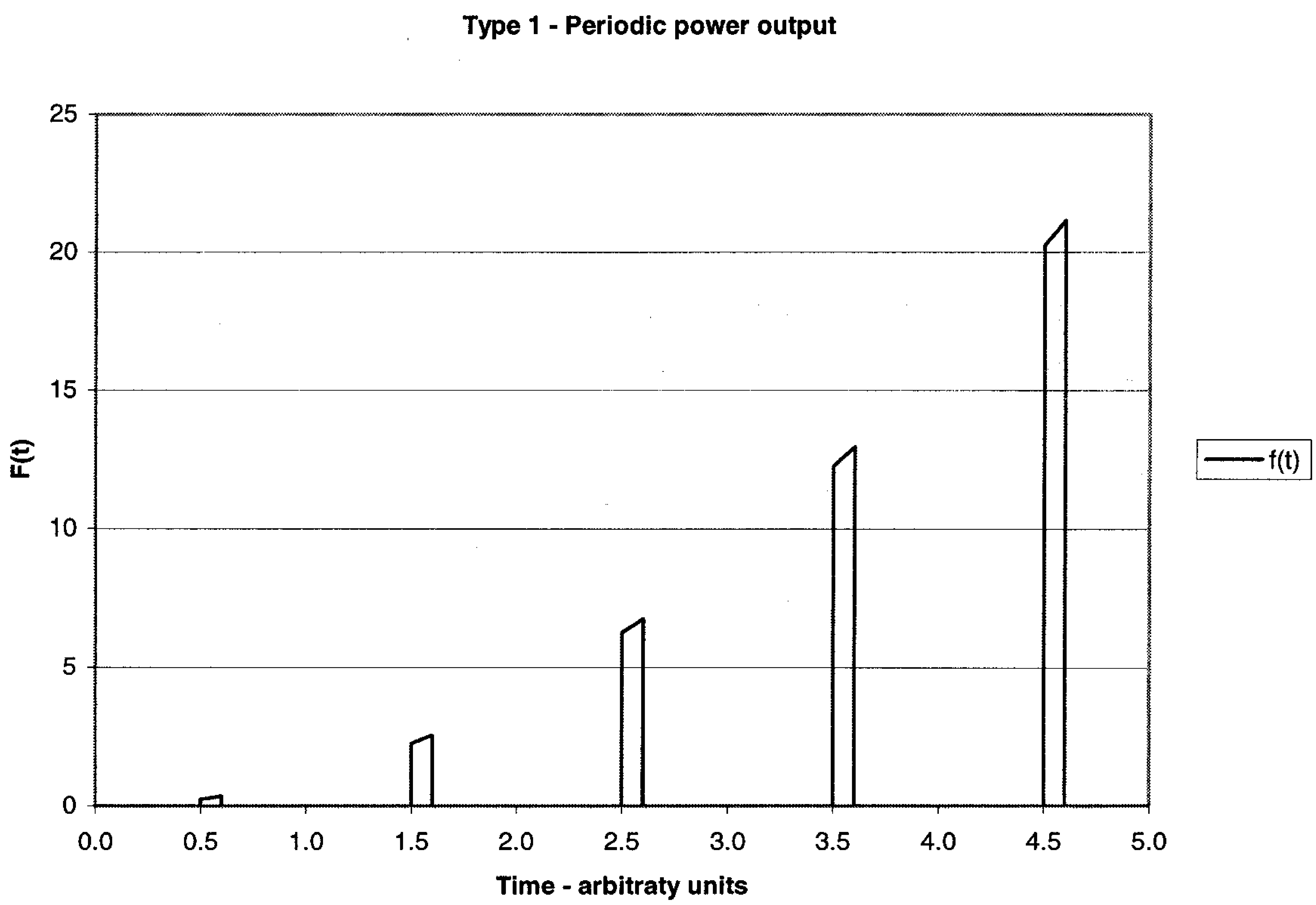
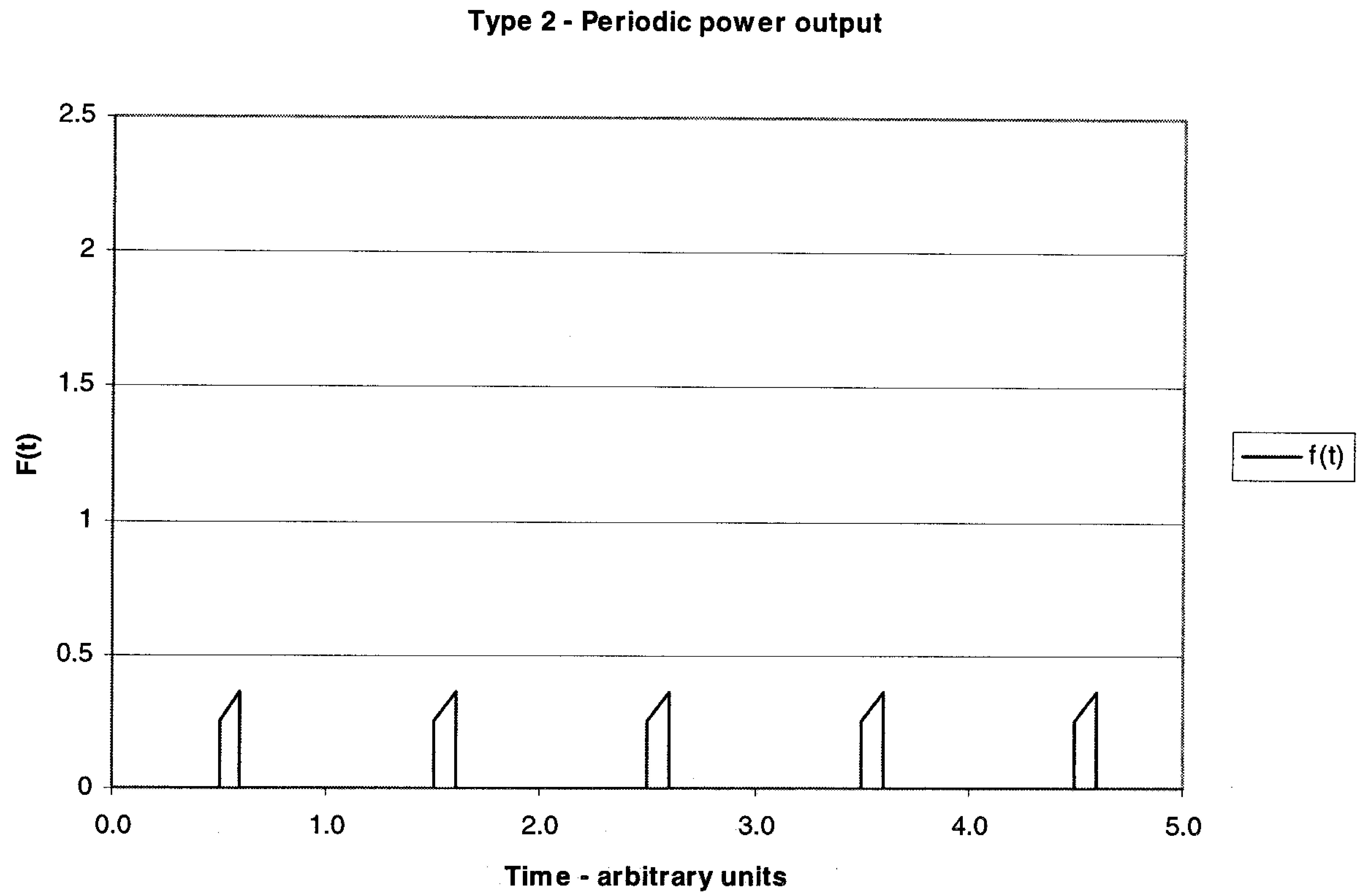


Figure 5 Example of a type 2 periodic function



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Appendix A

Conceptual Isomer Energy Supply Energy Balances

Case 1 - Constant Power, P

The value of local power consumption, $P_1(t)$, is:

P

The number of atoms, $N_{1atom}(t)$, required to produce the local power, can be determined from Equation (57). $N_{1atom}(t)$ is:

$$\frac{P\,\eta\text{trigger}}{\text{Etrigger}\,\beta\,\lambda}$$

The cumulative energy, $E_1(t)$, produced for local power at time t can be determined by integrating the power output over time, Equation (58). $E_1(t)$ is:

P t

The number of atoms required to supply the necessary feedback power, $N_{2atom}(t)$, can be determined by solving Equation (59). $N_{2atom}(t)$ is:

$$\frac{P\,\eta\text{trigger}}{-\text{Etrigger}\,\beta\,\lambda+\text{Etrigger}\,\beta^2\,\lambda}+e^{t\,(-1+\beta)\,\lambda}\,C[1]$$

The power output required to supply the necessary feedback power, $P_2(t)$, can be determined from Equation (60). $P_2(t)$ is:

$$\frac{\eta\text{trigger}\,(P\,\eta\text{trigger}+e^{t\,(-1+\beta)\,\lambda}\,\text{Etrigger}\,(-1+\beta)\,\beta\,\lambda\,C[1])}{\text{Etrigger}^2\,(-1+\beta)\,\beta^2\,\lambda^2}$$

The cumulative energy output required to supply the necessary feedback power, $E_2(t)$, can be determined from Equation (61). E2(t) is:

$$\frac{\eta\text{trigger}\,(P\,t\,\eta\text{trigger}+(-1+e^{t\,(-1+\beta)\,\lambda})\,\text{Etrigger}\,\beta\,C[1])}{\text{Etrigger}^2\,(-1+\beta)\,\beta^2\,\lambda^2}$$

The combined power output from $P_1(t)$ and $P_2(t)$ can be determined by adding the two terms. The sum of $P_1(t)$ and $P_2(t)$ is:

$$\frac{\eta\text{trigger}\,(P\,\eta\text{trigger}+e^{t\,(-1+\beta)\,\lambda}\,\text{Etrigger}\,(-1+\beta)\,\lambda\,C[1])}{\text{Etrigger}^2\,(-1+\beta)\,\beta\,\lambda^2}$$

The total number of atoms in the excited state at any time t can be determined by adding $N_{1atom}(t)$ and $N_{2atom}(t)$. The sum of $N_{1atom}(t)$ and $N_{2atom}(t)$ is:

$$\frac{P\,\eta\text{trigger}}{-\text{Etrigger}\,\lambda+\text{Etrigger}\,\beta\,\lambda}+e^{t\,(-1+\beta)\,\lambda}\,C[1]$$

The total energy output at any time t can be determined by adding $E_1(t)$ and $E_2(t)$. <u>The sum of $E_1(t)$ and $E_2(t)$ is</u>:

$$\frac{\eta\text{trigger}\ (\text{P t}\ \eta\text{trigger} + (-1 + e^{t\,(-1+\beta)\,\lambda})\ \text{Etrigger C[1]})}{\text{Etrigger}^2\ (-1+\beta)\ \beta\,\lambda^2}$$

The total amount of stored energy, $E_{store}(t)$, can be determined from the product of the number of atoms in the excited state and the energy released per decay. <u>$E_{store}(t)$ is</u>:

$$\text{Edecay}\ \eta\text{out}\left(\frac{\text{P}\ \eta\text{trigger}}{-\text{Etrigger}\ \lambda + \text{Etrigger}\ \beta\,\lambda} + e^{t\,(-1+\beta)\,\lambda}\ \text{C[1]}\right)$$

Case 2 - Power increases linearly with time P t

The value of local power consumption, $P_1(t)$, is:

$$P\ t$$

The number of atoms, $N_{1atom}(t)$, required to produce the local power, can be determined from Equation (57). $N_{1atom}(t)$ is:

$$\frac{P\ t\ \eta trigger}{Etrigger\ \beta\ \lambda}$$

The cumulative energy, $E_1(t)$, produced for local power at time t can be determined by integrating the power output over time, Equation (58). $E_1(t)$ is:

$$\frac{P\ t^2}{2}$$

The number of atoms required to supply the necessary feedback power, $N_{2atom}(t)$, can be determined by solving Equation (59). $N_{2atom}(t)$ is:

$$\frac{P\ \eta trigger\ (\beta + t\ (-1+\beta)\ \lambda)}{Etrigger\ (-1+\beta)^2\ \beta\ \lambda^2} + e^{t\ (-1+\beta)\ \lambda}\ C[1]$$

The power output required to supply the necessary feedback power, $P_2(t)$, can be determined from Equation (60). $P_2(t)$ is:

$$\frac{\eta trigger\ (P\ \eta trigger\ (\beta + t\ (-1+\beta)\ \lambda) + e^{t\ (-1+\beta)\ \lambda}\ Etrigger\ (-1+\beta)^2\ \beta\ \lambda^2\ C[1])}{Etrigger^2\ (-1+\beta)^2\ \beta^2\ \lambda^3}$$

The cumulative energy output required to supply the necessary feedback power, $E_2(t)$, can be determined from Equation (61). $E_2(t)$ is:

$$(\eta trigger\ (P\ t\ \eta trigger\ (2\ \beta + t\ (-1+\beta)\ \lambda) + 2\ (-1 + e^{t\ (-1+\beta)\ \lambda})\ Etrigger\ (-1+\beta)\ \beta\ \lambda\ C[1]))\ /\ (2\ Etrigger^2\ (-1+\beta)^2\ \beta^2\ \lambda^3)$$

The combined power output from $P_1(t)$ and $P_2(t)$ can be determined by adding the two terms. The sum of $P_1(t)$ and $P_2(t)$ is:

$$\frac{\eta trigger\ (P\ \eta trigger\ (1 + t\ (-1+\beta)\ \lambda) + e^{t\ (-1+\beta)\ \lambda}\ Etrigger\ (-1+\beta)^2\ \lambda^2\ C[1])}{Etrigger^2\ (-1+\beta)^2\ \beta\ \lambda^3}$$

The total number of atoms in the excited state at any time t can be determined by adding $N_{1atom}(t)$ and $N_{2atom}(t)$. The sum of $N_{1atom}(t)$ and $N_{2atom}(t)$ is:

$$\frac{P\,\eta trigger\,(1 + t\,(-1+\beta)\,\lambda) + e^{t\,(-1+\beta)\,\lambda}\,Etrigger\,(-1+\beta)^2\,\lambda^2\,C[1]}{Etrigger\,(-1+\beta)^2\,\lambda^2}$$

The total energy output at any time t can be determined by adding $E_1(t)$ and $E_2(t)$. The sum of $E_1(t)$ and $E_2(t)$ is:

$$(\eta trigger\,(P\,t\,\eta trigger\,(2 + t\,(-1+\beta)\,\lambda) + 2\,(-1 + e^{t\,(-1+\beta)\,\lambda})\,Etrigger\,(-1+\beta)\,\lambda\,C[1]))\,/\,(2\,Etrigger^2\,(-1+\beta)^2\,\beta\,\lambda^3)$$

The total amount of stored energy, $E_{store}(t)$, can be determined from the product of the number of atoms in the excited state and the energy released per decay. $E_{store}(t)$ is:

$$(Edecay\,\eta out\,(P\,\eta trigger\,(1 + t\,(-1+\beta)\,\lambda) + e^{t\,(-1+\beta)\,\lambda}\,Etrigger\,(-1+\beta)^2\,\lambda^2\,C[1]))\,/\,(Etrigger\,(-1+\beta)^2\,\lambda^2)$$

Case 3 - Power = P t^2

The value of local power consumption, $\underline{P_1(t)}$, is:

P t^2

The number of atoms, $N_{1atom}(t)$, required to produce the local power, can be determined from Equation (57). $\underline{N_{1atom}(t)}$ is:

$$\frac{P\, t^2\, \eta\mathrm{trigger}}{\mathrm{Etrigger}\, \beta\, \lambda}$$

The cumulative energy, $E_1(t)$, produced for local power at time t can be determined by integrating the power output over time, Equation (58). $\underline{E_1(t)}$ is:

$$\frac{P\, t^3}{3}$$

The number of atoms required to supply the necessary feedback power, $N_{2atom}(t)$, can be determined by solving Equation (59). $\underline{N_{2atom}(t)}$ is:

$$(P\, \eta\mathrm{trigger}\, (2\, \beta + 2\, t\, (-1+\beta)\, \beta\, \lambda + t^2\, (-1+\beta)^2\, \lambda^2) + e^{t\, (-1+\beta)\, \lambda}\, \mathrm{Etrigger}\, (-1+\beta)^3\, \beta\, \lambda^3\, C[1])\, /\, (\mathrm{Etrigger}\, (-1+\beta)^3\, \beta\, \lambda^3)$$

The power output required to supply the necessary feedback power, $P_2(t)$, can be determined from Equation (60). $\underline{P_2(t)}$ is:

$$(\eta\mathrm{trigger}\, (P\, \eta\mathrm{trigger}\, (2\, \beta + 2\, t\, (-1+\beta)\, \beta\, \lambda + t^2\, (-1+\beta)^2\, \lambda^2) + e^{t\, (-1+\beta)\, \lambda}\, \mathrm{Etrigger}\, (-1+\beta)^3\, \beta\, \lambda^3\, C[1]))\, /\, (\mathrm{Etrigger}^2\, (-1+\beta)^3\, \beta^2\, \lambda^4)$$

The cumulative energy output required to supply the necessary feedback power, $E_2(t)$, can be determined from Equation (61). $\underline{E_2(t)}$ is:

$$\left(\eta\mathrm{trigger}\, \left(P\, \eta\mathrm{trigger}\, \left(2\, t\, \beta + t^2\, (-1+\beta)\, \beta\, \lambda + \frac{1}{3}\, t^3\, (-1+\beta)^2\, \lambda^2\right) + (-1 + e^{t\, (-1+\beta)\, \lambda})\, \mathrm{Etrigger}\, (-1+\beta)^2\, \beta\, \lambda^2\, C[1]\right)\right) \Big/ (\mathrm{Etrigger}^2\, (-1+\beta)^3\, \beta^2\, \lambda^4)$$

The combined power output from $P_1(t)$ and $P_2(t)$ can be determined by adding the two terms. $\underline{\text{The sum of } P_1(t) \text{ and } P_2(t)}$ is:

$$(\eta\mathrm{trigger}\, (P\, \eta\mathrm{trigger}\, (2 + 2\, t\, (-1+\beta)\, \lambda + t^2\, (-1+\beta)^2\, \lambda^2) + e^{t\, (-1+\beta)\, \lambda}\, \mathrm{Etrigger}\, (-1+\beta)^3\, \lambda^3\, C[1]))\, /\, (\mathrm{Etrigger}^2\, (-1+\beta)^3\, \beta\, \lambda^4)$$

The total number of atoms in the excited state at any time t can be determined by adding $N_{1atom}(t)$ and $N_{2atom}(t)$. <u>The sum of $N_{1atom}(t)$ and $N_{2atom}(t)$ is</u>:

$$(P\,\eta trigger\,(2 + 2\,t\,(-1+\beta)\,\lambda + t^2\,(-1+\beta)^2\,\lambda^2) + e^{t\,(-1+\beta)\,\lambda}\,Etrigger\,(-1+\beta)^3\,\lambda^3\,C[1])\,/\,(Etrigger\,(-1+\beta)^3\,\lambda^3)$$

The total energy output at any time t can be determined by adding $E_1(t)$ and $E_2(t)$. <u>The sum of $E_1(t)$ and $E_2(t)$ is</u>:

$$\left(\eta trigger\,\left(\frac{1}{3}\,P\,t\,\eta trigger\,(6 + t\,(-1+\beta)\,\lambda\,(3 + t\,(-1+\beta)\,\lambda)) + (-1 + e^{t\,(-1+\beta)\,\lambda})\,Etrigger\,(-1+\beta)^2\,\lambda^2\,C[1]\right)\right) \Big/ (Etrigger^2\,(-1+\beta)^3\,\beta\,\lambda^4)$$

The total amount of stored energy, $E_{store}(t)$, can be determined from the product of the number of atoms in the excited state and the energy released per decay. <u>$E_{store}(t)$ is</u>:

$$(Edecay\,\eta out\,(P\,\eta trigger\,(2 + 2\,t\,(-1+\beta)\,\lambda + t^2\,(-1+\beta)^2\,\lambda^2) + e^{t\,(-1+\beta)\,\lambda}\,Etrigger\,(-1+\beta)^3\,\lambda^3\,C[1]))\,/\,(Etrigger\,(-1+\beta)^3\,\lambda^3)$$

Case 4 - Power =P t^3

The value of local power consumption, <u>$P_1(t)$, is</u>:

P t^3

The number of atoms, $N_{1atom}(t)$, required to produce the local power, can be determined from Equation (57). <u>$N_{1atom}(t)$ is</u>:

$$\frac{P\,t^3\,\eta\text{trigger}}{\text{Etrigger}\,\beta\,\lambda}$$

The cumulative energy, $E_1(t)$, produced for local power at time t can be determined by integrating the power output over time, Equation (58). <u>$E_1(t)$ is</u>:

$$\frac{P\,t^4}{4}$$

The number of atoms required to supply the necessary feedback power, $N_{2atom}(t)$, can be determined by solving Equation (59). <u>$N_{2atom}(t)$ is</u>:

$$\frac{P\,t^3\,\eta\text{trigger}}{-\text{Etrigger}\,\beta\,\lambda+\text{Etrigger}\,\beta^2\,\lambda}+\frac{3\,P\,\eta\text{trigger}\,(2+2\,t\,(-1+\beta)\,\lambda+t^2\,(-1+\beta)^2\,\lambda^2)}{\text{Etrigger}\,(-1+\beta)^4\,\lambda^4}+e^{t\,(-1+\beta)\,\lambda}\,C[1]$$

The power output required to supply the necessary feedback power, $P_2(t)$, can be determined from Equation (60). <u>$P_2(t)$ is</u>:

$$\frac{1}{\text{Etrigger}\,\beta\,\lambda}\left(\eta\text{trigger}\left(\frac{P\,t^3\,\eta\text{trigger}}{-\text{Etrigger}\,\beta\,\lambda+\text{Etrigger}\,\beta^2\,\lambda}+\frac{3\,P\,\eta\text{trigger}\,(2+2\,t\,(-1+\beta)\,\lambda+t^2\,(-1+\beta)^2\,\lambda^2)}{\text{Etrigger}\,(-1+\beta)^4\,\lambda^4}+e^{t\,(-1+\beta)\,\lambda}\,C[1]\right)\right)$$

The cumulative energy output required to supply the necessary feedback power, $E_2(t)$, can be determined from Equation (61). <u>$E_2(t)$</u> is:

$$\frac{1}{4\,\text{Etrigger}^2\,(-1+\beta)^4\,\beta^2\,\lambda^5}\,(\eta\text{trigger}\,(P\,t\,\eta\text{trigger}\,(24\,\beta+12\,t\,(-1+\beta)\,\beta\,\lambda+4\,t^2\,(-1+\beta)^2\,\beta\,\lambda^2+t^3\,(-1+\beta)^3\,\lambda^3)+4\,(-1+e^{t\,(-1+\beta)\,\lambda})\,\text{Etrigger}\,(-1+\beta)^3\,\beta\,\lambda^3\,C[1]))$$

The combined power output from $P_1(t)$ and $P_2(t)$ can be determined by adding the two terms. <u>The sum of $P_1(t)$ and $P_2(t)$ is</u>:

$$(\eta\text{trigger}\,(P\,\eta\text{trigger}\,(6+6\,t\,(-1+\beta)\,\lambda+3\,t^2\,(-1+\beta)^2\,\lambda^2+t^3\,(-1+\beta)^3\,\lambda^3)+e^{t\,(-1+\beta)\,\lambda}\,\text{Etrigger}\,(-1+\beta)^4\,\lambda^4\,C[1]))\,/\,(\text{Etrigger}^2\,(-1+\beta)^4\,\beta\,\lambda^5)$$

The total number of atoms in the excited state at any time t can be determined by adding $N_{1atom}(t)$ and $N_{2atom}(t)$. <u>The sum of $N_{1atom}(t)$ and $N_{2atom}(t)$ is</u>:

$$\frac{P\, t^3\, \eta trigger}{Etrigger\ \beta\, \lambda} + \frac{P\, t^3\, \eta trigger}{-Etrigger\ \beta\, \lambda + Etrigger\ \beta^2\, \lambda} + \frac{3\, P\, \eta trigger\ (2 + 2\, t\ (-1+\beta)\ \lambda + t^2\ (-1+\beta)^2\ \lambda^2)}{Etrigger\ (-1+\beta)^4\ \lambda^4} + e^{t\,(-1+\beta)\ \lambda}\ C[1]$$

The total energy output at any time t can be determined by adding $E_1(t)$ and $E_2(t)$. <u>The sum of $E_1(t)$ and $E_2(t)$ is</u>:

$$\frac{1}{Etrigger^2\ (-1+\beta)^4\ \beta\, \lambda^5} \Big(\eta trigger \Big(\frac{1}{4}\ P\, t\, \eta trigger\ (24 + t\ (-1+\beta)\ \lambda\ (12 + t\ (-1+\beta)\ \lambda\ (4 + t\ (-1+\beta)\ \lambda))) + (-1 + e^{t\,(-1+\beta)\ \lambda})\ Etrigger\ (-1+\beta)^3\ \lambda^3\ C[1]\Big)\Big)$$

The total amount of stored energy, $E_{store}(t)$, can be determined from the product of the number of atoms in the excited state and the energy released per decay. <u>$E_{store}(t)$ is</u>:

$$Edecay\ \eta out \left(\frac{P\, t^3\, \eta trigger}{Etrigger\ \beta\, \lambda} + \frac{P\, t^3\, \eta trigger}{-Etrigger\ \beta\, \lambda + Etrigger\ \beta^2\, \lambda} + \frac{3\, P\, \eta trigger\ (2 + 2\, t\ (-1+\beta)\ \lambda + t^2\ (-1+\beta)^2\ \lambda^2)}{Etrigger\ (-1+\beta)^4\ \lambda^4} + e^{t\,(-1+\beta)\ \lambda}\ C[1]\right)$$

Case 5 - Power =P Exp[θ t]

The value of local power consumption, P_1(t), is:

P Exp[θ t]

The number of atoms, N_{1atom}(t), required to produce the local power, can be determined from Equation (57). N_{1atom}(t) is:

$$\frac{e^{t\theta} P \,\eta\text{trigger}}{\text{Etrigger}\,\beta\,\lambda}$$

The cumulative energy, E_1(t), produced for local power at time t can be determined by integrating the power output over time, Equation (58). E_1(t) is:

$$\frac{(-1 + e^{t\theta})\,P}{\theta}$$

The number of atoms required to supply the necessary feedback power, N_{2atom}(t), can be determined by solving Equation (59). N_{2atom}(t) is:

$$e^{t(-1+\beta)\lambda}\left(-\frac{e^{t(\theta+\lambda-\beta\lambda)}\,P\,\eta\text{trigger}\,(\theta+\lambda)}{\text{Etrigger}\,\beta\,\lambda\,(\theta+\lambda-\beta\lambda)} + C[1]\right)$$

The power output required to supply the necessary feedback power, P_2(t), can be determined from Equation (60). P_2(t) is:

$$\frac{e^{t(-1+\beta)\lambda}\,\eta\text{trigger}\left(-\frac{e^{t(\theta+\lambda-\beta\lambda)}\,P\,\eta\text{trigger}\,(\theta+\lambda)}{\text{Etrigger}\,\beta\,\lambda\,(\theta+\lambda-\beta\lambda)} + C[1]\right)}{\text{Etrigger}\,\beta\,\lambda}$$

The cumulative energy output required to supply the necessary feedback power, E_2(t), can be determined from Equation (61). E_2(t) is:

$$(\eta\text{trigger}\,(-(-1 + e^{t\theta})\,P\,(-1+\beta)\,\eta\text{trigger}\,(\theta+\lambda) - (-1 + e^{t(-1+\beta)\lambda})\,\text{Etrigger}\,\beta\,\theta\,(-\theta-\lambda+\beta\lambda)\,C[1]))\,/\,(\text{Etrigger}^2\,(-1+\beta)\,\beta^2\,\theta\,\lambda^2\,(\theta+\lambda-\beta\lambda))$$

The combined power output from P_1(t) and P_2(t) can be determined by adding the two terms. The sum of P_1(t) and P_2(t) is:

$$-\frac{\eta\text{trigger}\,(e^{t\theta}\,P\,\eta\text{trigger} + e^{t(-1+\beta)\lambda}\,\text{Etrigger}\,(-\theta-\lambda+\beta\lambda)\,C[1])}{\text{Etrigger}^2\,\beta\,\lambda\,(\theta+\lambda-\beta\lambda)}$$

The total number of atoms in the excited state at any time t can be determined by adding N_{1atom}(t) and N_{2atom}(t). The sum of N_{1atom}(t) and N_{2atom}(t) is:

$$-\frac{e^{-t\lambda}\left(e^{t(\theta+\lambda)}\,P\,\eta\text{trigger}+e^{t\beta\lambda}\,\text{Etrigger}\,(-\theta-\lambda+\beta\lambda)\,C[1]\right)}{\text{Etrigger}\,(\theta+\lambda-\beta\lambda)}$$

The total energy output at any time t can be determined by adding $E_1(t)$ and $E_2(t)$. <u>The sum of $E_1(t)$ and $E_2(t)$ is</u>:

$$-(\eta\text{trigger}\,((-1+e^{t\theta})\,P\,(-1+\beta)\,\eta\text{trigger}\,\lambda-(-1+e^{t(-1+\beta)\lambda})\,\text{Etrigger}\,\theta\,(\theta+\lambda-\beta\lambda)\,C[1]))\,/\,(\text{Etrigger}^2\,(-1+\beta)\,\beta\,\theta\,\lambda^2\,(\theta+\lambda-\beta\lambda))$$

The total amount of stored energy, $E_{store}(t)$, can be determined from the product of the number of atoms in the excited state and the energy released per decay. <u>$E_{store}(t)$ is</u>:

$$-\frac{e^{-t\lambda}\,\text{Edecay}\,\eta\text{out}\,\left(e^{t(\theta+\lambda)}\,P\,\eta\text{trigger}+e^{t\beta\lambda}\,\text{Etrigger}\,(-\theta-\lambda+\beta\lambda)\,C[1]\right)}{\text{Etrigger}\,(\theta+\lambda-\beta\lambda)}$$

Case 6 - Power =P Sin[ω t]

The value of local power consumption, <u>P_1(t), is</u>:

P Sin[t ω]

The number of atoms, N_{1atom}(t), required to produce the local power, can be determined from Equation (57). <u>N_{1atom}(t) is</u>:

$$\frac{P\ \eta trigger\ Sin[t\ \omega]}{Etrigger\ \beta\ \lambda}$$

The cumulative energy, E_1(t), produced for local power at time t can be determined by integrating the power output over time, Equation (58). <u>E_1(t) is</u>:

$$\frac{P - P\ Cos[t\ \omega]}{\omega}$$

The number of atoms required to supply the necessary feedback power, N_{2atom}(t), can be determined by solving Equation (59). <u>N_{2atom}(t) is</u>:

$$(e^{t\ (-1+\beta)\ \lambda}\ Etrigger\ \beta\ \lambda\ ((-1+\beta)^2\ \lambda^2+\omega^2)\ C[1] + P\ \beta\ \eta trigger\ \lambda\ \omega\ Cos[t\ \omega] + P\ \eta trigger\ ((-1+\beta)\ \lambda^2-\omega^2)\ Sin[t\ \omega])\ /\ (Etrigger\ \beta\ \lambda\ ((-1+\beta)^2\ \lambda^2+\omega^2))$$

The power output required to supply the necessary feedback power, P_2(t), can be determined from Equation (60). <u>P_2(t) is</u>:

$$(\eta trigger\ (e^{t\ (-1+\beta)\ \lambda}\ Etrigger\ \beta\ \lambda\ ((-1+\beta)^2\ \lambda^2+\omega^2)\ C[1] + P\ \beta\ \eta trigger\ \lambda\ \omega\ Cos[t\ \omega] + P\ \eta trigger\ ((-1+\beta)\ \lambda^2-\omega^2)\ Sin[t\ \omega]))\ /\ (Etrigger^2\ \beta^2\ \lambda^2\ ((-1+\beta)^2\ \lambda^2+\omega^2))$$

The cumulative energy output required to supply the necessary feedback power, E_2(t), can be determined from Equation (61). <u>E_2(t)</u> is:

$$\left(\eta trigger\ \left(\frac{(-1+e^{t\ (-1+\beta)\ \lambda})\ Etrigger\ \beta\ ((-1+\beta)^2\ \lambda^2+\omega^2)\ C[1]}{-1+\beta} + \frac{P\ \eta trigger\ (-(-1+\beta)\ \lambda^2+\omega^2)\ (-1+Cos[t\ \omega])}{\omega} + P\ \beta\ \eta trigger\ \lambda\ Sin[t\ \omega]\right)\right) \Big/ (Etrigger^2\ \beta^2\ \lambda^2\ ((-1+\beta)^2\ \lambda^2+\omega^2))$$

The combined power output from P_1(t) and P_2(t) can be determined by adding the two terms. <u>The sum of P_1(t) and P_2(t) is</u>:

$$(\eta trigger\ (e^{t\ (-1+\beta)\ \lambda}\ Etrigger\ ((-1+\beta)^2\ \lambda^2+\omega^2)\ C[1] + P\ \eta trigger\ \omega\ Cos[t\ \omega] + P\ (-1+\beta)\ \eta trigger\ \lambda\ Sin[t\ \omega]))\ /\ (Etrigger^2\ \beta\ \lambda\ ((-1+\beta)^2\ \lambda^2+\omega^2))$$

The total number of atoms in the excited state at any time t can be determined by adding $N_{1atom}(t)$ and $N_{2atom}(t)$. <u>The sum of $N_{1atom}(t)$ and $N_{2atom}(t)$ is</u>:

$$(e^{t(-1+\beta)\lambda}\,\text{Etrigger}\,((-1+\beta)^2\lambda^2+\omega^2)\,C[1] + P\,\eta\text{trigger}\,\omega\,\text{Cos}[t\,\omega] + P\,(-1+\beta)\,\eta\text{trigger}\,\lambda\,\text{Sin}[t\,\omega])\,/\,(\text{Etrigger}\,((-1+\beta)^2\lambda^2+\omega^2))$$

The total energy output at any time t can be determined by adding $E_1(t)$ and $E_2(t)$. <u>The sum of $E_1(t)$ and $E_2(t)$ is</u>:

$$\left(\eta\text{trigger}\left(\frac{(-1+e^{t(-1+\beta)\lambda})\,\text{Etrigger}\,((-1+\beta)^2\lambda^2+\omega^2)\,C[1]}{(-1+\beta)\,\lambda} - \frac{P\,(-1+\beta)\,\eta\text{trigger}\,\lambda\,(-1+\text{Cos}[t\,\omega])}{\omega} + P\,\eta\text{trigger}\,\text{Sin}[t\,\omega]\right)\right)\Big/ (\text{Etrigger}^2\,\beta\,\lambda\,((-1+\beta)^2\lambda^2+\omega^2))$$

The total amount of stored energy, $E_{store}(t)$, can be determined from the product of the number of atoms in the excited state and the energy released per decay. <u>$E_{store}(t)$ is</u>:

$$(\text{Edecay}\,\eta\text{out}\,(e^{t(-1+\beta)\lambda}\,\text{Etrigger}\,((-1+\beta)^2\lambda^2+\omega^2)\,C[1] + P\,\eta\text{trigger}\,\omega\,\text{Cos}[t\,\omega] + P\,(-1+\beta)\,\eta\text{trigger}\,\lambda\,\text{Sin}[t\,\omega]))\,/\,(\text{Etrigger}\,((-1+\beta)^2\lambda^2+\omega^2))$$

Case 7 - Power =P Cos[ω t]

The value of local power consumption, <u>P_1(t), is</u>:

$$P \,\mathrm{Cos}[t\,\omega]$$

The number of atoms, N_{1atom}(t), required to produce the local power, can be determined from Equation (57). <u>N_{1atom}(t) is</u>:

$$\frac{P\,\eta\mathrm{trigger}\,\mathrm{Cos}[t\,\omega]}{\mathrm{Etrigger}\,\beta\,\lambda}$$

The cumulative energy, E_1(t), produced for local power at time t can be determined by integrating the power output over time, Equation (58). <u>E_1(t) is</u>:

$$\frac{P\,\mathrm{Sin}[t\,\omega]}{\omega}$$

The number of atoms required to supply the necessary feedback power, N_{2atom}(t), can be determined by solving Equation (59). <u>N_{2atom}(t) is</u>:

$$e^{t\,(-1+\beta)\,\lambda}\,C[1] + \frac{P\,\eta\mathrm{trigger}\,\left(\left((-1+\beta)\,\lambda^2-\omega^2\right)\mathrm{Cos}[t\,\omega] - \beta\,\lambda\,\omega\,\mathrm{Sin}[t\,\omega]\right)}{\mathrm{Etrigger}\,\beta\,\lambda\,\left((-1+\beta)^2\,\lambda^2+\omega^2\right)}$$

The power output required to supply the necessary feedback power, P_2(t), can be determined from Equation (60). <u>P_2(t) is</u>:

$$\frac{\eta\mathrm{trigger}\,\left(e^{t\,(-1+\beta)\,\lambda}\,C[1] + \frac{P\,\eta\mathrm{trigger}\,\left(\left((-1+\beta)\,\lambda^2-\omega^2\right)\mathrm{Cos}[t\,\omega]-\beta\,\lambda\,\omega\,\mathrm{Sin}[t\,\omega]\right)}{\mathrm{Etrigger}\,\beta\,\lambda\,\left((-1+\beta)^2\,\lambda^2+\omega^2\right)}\right)}{\mathrm{Etrigger}\,\beta\,\lambda}$$

The cumulative energy output required to supply the necessary feedback power, E_2(t), can be determined from Equation (61). <u>E_2(t)</u> is:

$$\begin{aligned}&(\eta\mathrm{trigger}\,(\beta\,\omega\,(-P\,(-1+\beta)\,\eta\mathrm{trigger}\,\lambda + \\ &\qquad (-1+e^{t\,(-1+\beta)\,\lambda})\,\mathrm{Etrigger}\,((-1+\beta)^2\,\lambda^2+\omega^2)\,C[1]) + \\ &\quad P\,(-1+\beta)\,\eta\mathrm{trigger}\,(\beta\,\lambda\,\omega\,\mathrm{Cos}[t\,\omega] + ((-1+\beta)\,\lambda^2-\omega^2)\,\mathrm{Sin}[t\,\omega]))) \,/ \\ &(\mathrm{Etrigger}^2\,(-1+\beta)\,\beta^2\,\lambda^2\,\omega\,((-1+\beta)^2\,\lambda^2+\omega^2))\end{aligned}$$

The combined power output from P_1(t) and P_2(t) can be determined by adding the two terms. <u>The sum of P_1(t) and P_2(t) is</u>:

$$(\eta trigger\ (e^{t\,(-1+\beta)\,\lambda}\ Etrigger\ ((-1+\beta)^2\lambda^2+\omega^2)\ C[1] + P\,(-1+\beta)\ \eta trigger\ \lambda\,Cos[t\,\omega] - P\ \eta trigger\ \omega\,Sin[t\,\omega]))\ /\ (Etrigger^2\,\beta\,\lambda\,((-1+\beta)^2\lambda^2+\omega^2))$$

The total number of atoms in the excited state at any time t can be determined by adding $N_{1atom}(t)$ and $N_{2atom}(t)$. The sum of $N_{1atom}(t)$ and $N_{2atom}(t)$ is:

$$(e^{t\,(-1+\beta)\,\lambda}\ Etrigger\ ((-1+\beta)^2\lambda^2+\omega^2)\ C[1] + P\,(-1+\beta)\ \eta trigger\ \lambda\,Cos[t\,\omega] - P\ \eta trigger\ \omega\,Sin[t\,\omega])\ /\ (Etrigger\ ((-1+\beta)^2\lambda^2+\omega^2))$$

The total energy output at any time t can be determined by adding $E_1(t)$ and $E_2(t)$. The sum of $E_1(t)$ and $E_2(t)$ is:

$$\left(\eta trigger\left(\frac{(-1+e^{t\,(-1+\beta)\,\lambda})\ Etrigger\ ((-1+\beta)^2\lambda^2+\omega^2)\ C[1]}{(-1+\beta)\,\lambda} + P\ \eta trigger\ (-1+Cos[t\,\omega]) + \frac{P\,(-1+\beta)\ \eta trigger\ \lambda\,Sin[t\,\omega]}{\omega}\right)\right)\Big/\ (Etrigger^2\,\beta\,\lambda\,((-1+\beta)^2\lambda^2+\omega^2))$$

The total amount of stored energy, $E_{store}(t)$, can be determined from the product of the number of atoms in the excited state and the energy released per decay. $E_{store}(t)$ is:

$$(Edecay\ \eta out\ (e^{t\,(-1+\beta)\,\lambda}\ Etrigger\ ((-1+\beta)^2\lambda^2+\omega^2)\ C[1] + P\,(-1+\beta)\ \eta trigger\ \lambda\,Cos[t\,\omega] - P\ \eta trigger\ \omega\,Sin[t\,\omega]))\ /\ (Etrigger\ ((-1+\beta)^2\lambda^2+\omega^2))$$

Case 8 - Power =P Sin[ω t+ θ]

The value of local power consumption, $\underline{P_1(t)}$, is:

$$P \sin[\theta + t\,\omega]$$

The number of atoms, $N_{1atom}(t)$, required to produce the local power, can be determined from Equation (57). $\underline{N_{1atom}(t)}$ is:

$$\frac{P\,\eta trigger\,\sin[\theta + t\,\omega]}{Etrigger\,\beta\,\lambda}$$

The cumulative energy, $E_1(t)$, produced for local power at time t can be determined by integrating the power output over time, Equation (58). $\underline{E_1(t)}$ is:

$$\frac{P\,(\cos[\theta] - \cos[\theta + t\,\omega])}{\omega}$$

The number of atoms required to supply the necessary feedback power, $N_{2atom}(t)$, can be determined by solving Equation (59). $\underline{N_{2atom}(t)}$ is:

$$\left(e^{t\,(-1+\beta)\,\lambda}\,Etrigger\,\beta\,\lambda\,((-1+\beta)^2\,\lambda^2 + \omega^2)\,C[1] + P\,\beta\,\eta trigger\,\lambda\,\omega\,\cos[\theta + t\,\omega] + P\,\eta trigger\,((-1+\beta)\,\lambda^2 - \omega^2)\,\sin[\theta + t\,\omega]\right) / \left(Etrigger\,\beta\,\lambda\,((-1+\beta)^2\,\lambda^2 + \omega^2)\right)$$

The power output required to supply the necessary feedback power, $P_2(t)$, can be determined from Equation (60). $\underline{P_2(t)}$ is:

$$\left(\eta trigger\,\left(e^{t\,(-1+\beta)\,\lambda}\,Etrigger\,\beta\,\lambda\,((-1+\beta)^2\,\lambda^2 + \omega^2)\,C[1] + P\,\beta\,\eta trigger\,\lambda\,\omega\,\cos[\theta + t\,\omega] + P\,\eta trigger\,((-1+\beta)\,\lambda^2 - \omega^2)\,\sin[\theta + t\,\omega]\right)\right) / \left(Etrigger^2\,\beta^2\,\lambda^2\,((-1+\beta)^2\,\lambda^2 + \omega^2)\right)$$

The cumulative energy output required to supply the necessary feedback power, $E_2(t)$, can be determined from Equation (61). $\underline{E_2(t)}$ is:

$$\left(\eta trigger\,\left(\frac{(-1 + e^{t\,(-1+\beta)\,\lambda})\,Etrigger\,\beta\,((-1+\beta)^2\,\lambda^2 + \omega^2)\,C[1]}{-1+\beta} + \frac{P\,\eta trigger\,(-(-1+\beta)\,\lambda^2 + \omega^2)\,(-\cos[\theta] + \cos[\theta + t\,\omega])}{\omega} + P\,\beta\,\eta trigger\,\lambda\,(-\sin[\theta] + \sin[\theta + t\,\omega])\right)\right) \Big/ \left(Etrigger^2\,\beta^2\,\lambda^2\,((-1+\beta)^2\,\lambda^2 + \omega^2)\right)$$

The combined power output from $P_1(t)$ and $P_2(t)$ can be determined by adding the two terms. The sum of $P_1(t)$ and $P_2(t)$ is:

$$(\eta\text{trigger}\ (e^{t\,(-1+\beta)\,\lambda}\ \text{Etrigger}\ ((-1+\beta)^2\,\lambda^2+\omega^2)\ C[1] + P\,\eta\text{trigger}\ \omega\,\text{Cos}[\theta+t\,\omega] + P\,(-1+\beta)\ \eta\text{trigger}\ \lambda\,\text{Sin}[\theta+t\,\omega]))\,/\,(\text{Etrigger}^2\,\beta\,\lambda\,((-1+\beta)^2\,\lambda^2+\omega^2))$$

The total number of atoms in the excited state at any time t can be determined by adding $N_{1atom}(t)$ and $N_{2atom}(t)$. <u>The sum of $N_{1atom}(t)$ and $N_{2atom}(t)$ is</u>:

$$(e^{t\,(-1+\beta)\,\lambda}\ \text{Etrigger}\ ((-1+\beta)^2\,\lambda^2+\omega^2)\ C[1] + P\,\eta\text{trigger}\ \omega\,\text{Cos}[\theta+t\,\omega] + P\,(-1+\beta)\ \eta\text{trigger}\ \lambda\,\text{Sin}[\theta+t\,\omega])\,/\,(\text{Etrigger}\ ((-1+\beta)^2\,\lambda^2+\omega^2))$$

The total energy output at any time t can be determined by adding $E_1(t)$ and $E_2(t)$. <u>The sum of $E_1(t)$ and $E_2(t)$ is</u>:

$$\left(\eta\text{trigger}\left(\frac{(-1+e^{t\,(-1+\beta)\,\lambda})\ \text{Etrigger}\ ((-1+\beta)^2\,\lambda^2+\omega^2)\ C[1]}{(-1+\beta)\,\lambda} + \frac{P\,(-1+\beta)\ \eta\text{trigger}\ \lambda\,(\text{Cos}[\theta]-\text{Cos}[\theta+t\,\omega])}{\omega} + P\,\eta\text{trigger}\ (-\text{Sin}[\theta]+\text{Sin}[\theta+t\,\omega])\right)\right)\Big/\,(\text{Etrigger}^2\,\beta\,\lambda\,((-1+\beta)^2\,\lambda^2+\omega^2))$$

The total amount of stored energy, $E_{store}(t)$, can be determined from the product of the number of atoms in the excited state and the energy released per decay. <u>$E_{store}(t)$ is</u>:

$$\frac{1}{\text{Etrigger}\ ((-1+\beta)^2\,\lambda^2+\omega^2)}\,(\text{Edecay}\ \eta\text{out}\ (e^{t\,(-1+\beta)\,\lambda}\ \text{Etrigger}\ ((-1+\beta)^2\,\lambda^2+\omega^2)\ C[1] + P\,\eta\text{trigger}\ \omega\,\text{Cos}[\theta+t\,\omega] + P\,(-1+\beta)\ \eta\text{trigger}\ \lambda\,\text{Sin}[\theta+t\,\omega]))$$

Case 9 - Power =P Sin[ω t+ θ] + K

The value of local power consumption, $P_1(t)$, is:

K+P Sin[θ+t ω]

The number of atoms, $N_{1atom}(t)$, required to produce the local power, can be determined from Equation (57). $N_{1atom}(t)$ is:

$$\frac{\eta\text{trigger}\ (K+P\,\text{Sin}[\theta+t\,\omega])}{\text{Etrigger}\ \beta\,\lambda}$$

The cumulative energy, $E_1(t)$, produced for local power at time t can be determined by integrating the power output over time, Equation (58). $E_1(t)$ is:

$$K\,t+\frac{P\ (\text{Cos}[\theta]-\text{Cos}[\theta+t\,\omega])}{\omega}$$

The number of atoms required to supply the necessary feedback power, $N_{2atom}(t)$, can be determined by solving Equation (59). $N_{2atom}(t)$ is:

$$\frac{K\,\eta\text{trigger}}{-\text{Etrigger}\ \beta\,\lambda+\text{Etrigger}\ \beta^2\,\lambda}+e^{t\,(-1+\beta)\,\lambda}\,C[1]+$$
$$\frac{P\,\eta\text{trigger}\ (\beta\,\lambda\,\omega\,\text{Cos}[\theta+t\,\omega]+((-1+\beta)\,\lambda^2-\omega^2)\ \text{Sin}[\theta+t\,\omega])}{\text{Etrigger}\ \beta\,\lambda\ ((-1+\beta)^2\,\lambda^2+\omega^2)}$$

The power output required to supply the necessary feedback power, $P_2(t)$, can be determined from Equation (60). P2(t) is:

$$\frac{1}{\text{Etrigger}\ \beta\,\lambda}\left(\eta\text{trigger}\ \left(\frac{K\,\eta\text{trigger}}{-\text{Etrigger}\ \beta\,\lambda+\text{Etrigger}\ \beta^2\,\lambda}+e^{t\,(-1+\beta)\,\lambda}\,C[1]+\right.\right.$$
$$\left.\left.\frac{P\,\eta\text{trigger}\ (\beta\,\lambda\,\omega\,\text{Cos}[\theta+t\,\omega]+((-1+\beta)\,\lambda^2-\omega^2)\ \text{Sin}[\theta+t\,\omega])}{\text{Etrigger}\ \beta\,\lambda\ ((-1+\beta)^2\,\lambda^2+\omega^2)}\right)\right)$$

The cumulative energy output required to supply the necessary feedback power, $E_2(t)$, can be determined from Equation (61). E2(t) is:

$$\Big(\eta\text{trigger}$$
$$\Big(\omega\ ((-1+\beta)^2\,\lambda^2+\omega^2)\ (K\,t\,\eta\text{trigger}+(-1+e^{t\,(-1+\beta)\,\lambda})\ \text{Etrigger}\ \beta\,C[1])+$$
$$2\,P\ (-1+\beta)\ \eta\text{trigger}\ \text{Sin}\left[\frac{t\,\omega}{2}\right]$$
$$\left(\beta\,\lambda\,\omega\,\text{Cos}\left[\theta+\frac{t\,\omega}{2}\right]+((-1+\beta)\,\lambda^2-\omega^2)\ \text{Sin}\left[\theta+\frac{t\,\omega}{2}\right]\right)\Big)\Big)\Big/$$
$$(\text{Etrigger}^2\ (-1+\beta)\ \beta^2\,\lambda^2\,\omega\ ((-1+\beta)^2\,\lambda^2+\omega^2))$$

The combined power output from $P_1(t)$ and $P_2(t)$ can be determined by adding the two terms. The sum of $P_1(t)$ and $P_2(t)$ is:

$$(\eta\text{trigger}\ (((-1+\beta)^2\lambda^2+\omega^2)\ (K\,\eta\text{trigger}+e^{t\,(-1+\beta)\,\lambda}\,\text{Etrigger}\ (-1+\beta)\,\lambda\,C[1])+ P\,(-1+\beta)\,\eta\text{trigger}\,\lambda\,\omega\,\text{Cos}[\theta+t\,\omega]+ P\,(-1+\beta)^2\,\eta\text{trigger}\,\lambda^2\,\text{Sin}[\theta+t\,\omega]))\,/\,(\text{Etrigger}^2\,(-1+\beta)\,\beta\,\lambda^2\,((-1+\beta)^2\lambda^2+\omega^2))$$

The total number of atoms in the excited state at any time t can be determined by adding $N_{1atom}(t)$ and $N_{2atom}(t)$. The sum of $N_{1atom}(t)$ and $N_{2atom}(t)$ is:

$$\left(P\,\eta\text{trigger}\,\omega\,\text{Cos}[\theta+t\,\omega]+\frac{1}{(-1+\beta)\,\lambda}(((-1+\beta)^2\lambda^2+\omega^2)\,(K\,\eta\text{trigger}+e^{t\,(-1+\beta)\,\lambda}\,\text{Etrigger}\,(-1+\beta)\,\lambda\,C[1])+P\,(-1+\beta)^2\,\eta\text{trigger}\,\lambda^2\,\text{Sin}[\theta+t\,\omega])\right)\Big/(\text{Etrigger}\,((-1+\beta)^2\lambda^2+\omega^2))$$

The total energy output at any time t can be determined by adding $E_1(t)$ and $E_2(t)$. The sum of $E_1(t)$ and $E_2(t)$ is:

$$\left(\eta\text{trigger}\left(((-1+\beta)^2\lambda^2+\omega^2)\,(K\,t\,\eta\text{trigger}+(-1+e^{t\,(-1+\beta)\,\lambda})\,\text{Etrigger}\,C[1])+\frac{2\,P\,(-1+\beta)\,\eta\text{trigger}\,\lambda\,\text{Sin}[\frac{t\omega}{2}]\,(\omega\,\text{Cos}[\theta+\frac{t\omega}{2}]+(-1+\beta)\,\lambda\,\text{Sin}[\theta+\frac{t\omega}{2}])}{\omega}\right)\right)\Big/(\text{Etrigger}^2\,(-1+\beta)\,\beta\,\lambda^2\,((-1+\beta)^2\lambda^2+\omega^2))$$

The total amount of stored energy, $E_{store}(t)$, can be determined from the product of the number of atoms in the excited state and the energy released per decay. $E_{store}(t)$ is:

$$\frac{1}{\text{Etrigger}\,((-1+\beta)^2\lambda^2+\omega^2)}\left(\text{Edecay}\,\eta\text{out}\left(P\,\eta\text{trigger}\,\omega\,\text{Cos}[\theta+t\,\omega]+\frac{1}{(-1+\beta)\,\lambda}(((-1+\beta)^2\lambda^2+\omega^2)\,(K\,\eta\text{trigger}+e^{t\,(-1+\beta)\,\lambda}\,\text{Etrigger}\,(-1+\beta)\,\lambda\,C[1])+P\,(-1+\beta)^2\,\eta\text{trigger}\,\lambda^2\,\text{Sin}[\theta+t\,\omega])\right)\right)$$

Case 10 - Power =P Cos[ω t+ θ] + K

The value of local power consumption, P_1(t), is:

$$K+P\,Cos[\theta+t\,\omega]$$

The number of atoms, N_{1atom}(t), required to produce the local power, can be determined from Equation (57). N_{1atom}(t) is:

$$\frac{\eta trigger\ (K+P\,Cos[\theta+t\,\omega])}{Etrigger\ \beta\,\lambda}$$

The cumulative energy, E_1(t), produced for local power at time t can be determined by integrating the power output over time, Equation (58). E_1(t) is:

$$K\,t+\frac{P\,(-Sin[\theta]+Sin[\theta+t\,\omega])}{\omega}$$

The number of atoms required to supply the necessary feedback power, N_{2atom}(t), can be determined by solving Equation (59). N_{2atom}(t) is:

$$\frac{K\,\eta trigger}{-Etrigger\ \beta\,\lambda+Etrigger\ \beta^2\,\lambda}+e^{t\,(-1+\beta)\,\lambda}\,C[1]+$$
$$\frac{P\,\eta trigger\ (((-1+\beta)\,\lambda^2-\omega^2)\,Cos[\theta+t\,\omega]-\beta\,\lambda\,\omega\,Sin[\theta+t\,\omega])}{Etrigger\ \beta\,\lambda\,((-1+\beta)^2\,\lambda^2+\omega^2)}$$

The power output required to supply the necessary feedback power, P_2(t), can be determined from Equation (60). P2(t) is:

$$\frac{1}{Etrigger\ \beta\,\lambda}\left(\eta trigger\left(\frac{K\,\eta trigger}{-Etrigger\ \beta\,\lambda+Etrigger\ \beta^2\,\lambda}+e^{t\,(-1+\beta)\,\lambda}\,C[1]+\right.\right.$$
$$\left.\left.\frac{P\,\eta trigger\ (((-1+\beta)\,\lambda^2-\omega^2)\,Cos[\theta+t\,\omega]-\beta\,\lambda\,\omega\,Sin[\theta+t\,\omega])}{Etrigger\ \beta\,\lambda\,((-1+\beta)^2\,\lambda^2+\omega^2)}\right)\right)$$

The cumulative energy output required to supply the necessary feedback power, E_2(t), can be determined from Equation (61). E2(t) is:

$$\Big(\eta trigger$$
$$\Big(\omega\,((-1+\beta)^2\,\lambda^2+\omega^2)\,(K\,t\,\eta trigger+(-1+e^{t\,(-1+\beta)\,\lambda})\,Etrigger\ \beta\,C[1])-$$
$$2\,P\,(-1+\beta)\,\eta trigger\ Sin\left[\frac{t\,\omega}{2}\right]$$
$$\left((-(-1+\beta)\,\lambda^2+\omega^2)\,Cos\left[\theta+\frac{t\,\omega}{2}\right]+\beta\,\lambda\,\omega\,Sin\left[\theta+\frac{t\,\omega}{2}\right]\right)\Big)\Big)\Big/$$
$$(Etrigger^2\,(-1+\beta)\,\beta^2\,\lambda^2\,\omega\,((-1+\beta)^2\,\lambda^2+\omega^2))$$

The combined power output from P_1(t) and P_2(t) can be determined by adding the two terms. The sum of P_1(t) and P_2(t) is:

$$(\eta\text{trigger}\ (((-1+\beta)^2\lambda^2+\omega^2)\ (\text{K}\,\eta\text{trigger} + e^{t\,(-1+\beta)\,\lambda}\,\text{Etrigger}\,(-1+\beta)\,\lambda\,\text{C}[1]) + \text{P}\,(-1+\beta)^2\,\eta\text{trigger}\,\lambda^2\,\text{Cos}[\theta+t\,\omega] - \text{P}\,(-1+\beta)\,\eta\text{trigger}\,\lambda\,\omega\,\text{Sin}[\theta+t\,\omega]))\ /\ (\text{Etrigger}^2\,(-1+\beta)\,\beta\,\lambda^2\,((-1+\beta)^2\lambda^2+\omega^2))$$

The total number of atoms in the excited state at any time t can be determined by adding $N_{1atom}(t)$ and $N_{2atom}(t)$. <u>The sum of $N_{1atom}(t)$ and $N_{2atom}(t)$ is</u>:

$$\frac{\text{K}\,\eta\text{trigger}}{\text{Etrigger}\,\beta\,\lambda} + \frac{\text{K}\,\eta\text{trigger}}{-\text{Etrigger}\,\beta\,\lambda + \text{Etrigger}\,\beta^2\,\lambda} + e^{t\,(-1+\beta)\,\lambda}\,\text{C}[1] + \frac{\text{P}\,(-1+\beta)\,\eta\text{trigger}\,\lambda\,\text{Cos}[\theta+t\,\omega]}{\text{Etrigger}\,((-1+\beta)^2\lambda^2+\omega^2)} - \frac{\text{P}\,\eta\text{trigger}\,\omega\,\text{Sin}[\theta+t\,\omega]}{\text{Etrigger}\,((-1+\beta)^2\lambda^2+\omega^2)}$$

The total energy output at any time t can be determined by adding $E_1(t)$ and $E_2(t)$. <u>The sum of $E_1(t)$ and $E_2(t)$ is</u>:

$$\left(\eta\text{trigger}\left(((-1+\beta)^2\lambda^2+\omega^2)\ (\text{K}\,t\,\eta\text{trigger} + (-1+e^{t\,(-1+\beta)\,\lambda})\,\text{Etrigger}\,\text{C}[1]) + \frac{1}{\omega}\,(\text{P}\,(-1+\beta)\,\eta\text{trigger}\,\lambda\,(-\omega\,\text{Cos}[\theta] + \omega\,\text{Cos}[\theta+t\,\omega] + (-1+\beta)\,\lambda\,(-\text{Sin}[\theta]+\text{Sin}[\theta+t\,\omega])))\right)\right) \Big/ (\text{Etrigger}^2\,(-1+\beta)\,\beta\,\lambda^2\,((-1+\beta)^2\lambda^2+\omega^2))$$

The total amount of stored energy, $E_{store}(t)$, can be determined from the product of the number of atoms in the excited state and the energy released per decay. <u>$E_{store}(t)$ is</u>:

$$\text{Edecay}\,\eta\text{out}\left(\frac{\text{K}\,\eta\text{trigger}}{\text{Etrigger}\,\beta\,\lambda} + \frac{\text{K}\,\eta\text{trigger}}{-\text{Etrigger}\,\beta\,\lambda + \text{Etrigger}\,\beta^2\,\lambda} + e^{t\,(-1+\beta)\,\lambda}\,\text{C}[1] + \frac{\text{P}\,(-1+\beta)\,\eta\text{trigger}\,\lambda\,\text{Cos}[\theta+t\,\omega]}{\text{Etrigger}\,((-1+\beta)^2\lambda^2+\omega^2)} - \frac{\text{P}\,\eta\text{trigger}\,\omega\,\text{Sin}[\theta+t\,\omega]}{\text{Etrigger}\,((-1+\beta)^2\lambda^2+\omega^2)}\right)$$

Case 11 - Power = P t^n -- note N2atom is not defined for exponents < 1

The value of local power consumption, <u>$P_1(t)$, is</u>:

P t^n

The number of atoms, $N_{1atom}(t)$, required to produce the local power, can be determined from Equation (57). <u>$N_{1atom}(t)$ is</u>:

$$\frac{P\, t^n\, \eta\text{trigger}}{\text{Etrigger}\, \beta\, \lambda}$$

The cumulative energy, $E_1(t)$, produced for local power at time t can be determined by integrating the power output over time, Equation (58). <u>$E_1(t)$ is</u>:

P t^(n+1)/(n+1)

The number of atoms required to supply the necessary feedback power, $N_{2atom}(t)$, can be determined by solving Equation (59). <u>$N_{2atom}(t)$ is</u>:

$$\frac{1}{\text{Etrigger}\, \beta\, \lambda} \left(e^{t\,(-1+\beta)\,\lambda} \left(\text{Etrigger}\, \beta\, \lambda\, C[1] + P\, t^n\, \eta\text{trigger}\, \left(n\, \text{ExpIntegralE}[1-n,\, t\,(-1+\beta)\,\lambda] + t\, \lambda\, \text{ExpIntegralE}[-n,\, t\,(-1+\beta)\,\lambda]\right)\right)\right)$$

The power output required to supply the necessary feedback power, $P_2(t)$, can be determined from Equation (60). <u>$P_2(t)$ is</u>:

$$\frac{1}{\text{Etrigger}^2\, \beta^2\, \lambda^2} \left(e^{t\,(-1+\beta)\,\lambda}\, \eta\text{trigger} \left(\text{Etrigger}\, \beta\, \lambda\, C[1] + P\, t^n\, \eta\text{trigger}\, \left(n\, \text{ExpIntegralE}[1-n,\, t\,(-1+\beta)\,\lambda] + t\, \lambda\, \text{ExpIntegralE}[-n,\, t\,(-1+\beta)\,\lambda]\right)\right)\right)$$

The cumulative energy output required to supply the necessary feedback power, $E_2(t)$, can be determined from Equation (61). <u>E2(t)</u> is:

$$\int_0^t \frac{1}{\text{Etrigger}^2\, \beta^2\, \lambda^2} \left(e^{t\,(-1+\beta)\,\lambda}\, \eta\text{trigger} \left(\text{Etrigger}\, \beta\, \lambda\, C[1] + P\, t^n\, \eta\text{trigger}\, \left(n\, \text{ExpIntegralE}[1-n,\, t\,(-1+\beta)\,\lambda] + t\, \lambda\, \text{ExpIntegralE}[-n,\, t\,(-1+\beta)\,\lambda]\right)\right)\right) dt$$

The combined power output from $P_1(t)$ and $P_2(t)$ can be determined by adding the two terms. <u>The sum of $P_1(t)$ and $P_2(t)$ is</u>:

$$\frac{1}{\text{Etrigger}^2\,\beta^2\,\lambda^2}\left(\eta\text{trigger}\left(P\,t^n\,\eta\text{trigger}+e^{t(-1+\beta)\lambda}\,\text{Etrigger}\,\beta\,\lambda\,C[1]+e^{t(-1+\beta)\lambda}\,n\,P\,t^n\,\eta\text{trigger}\;\text{ExpIntegralE}[1-n,\,t(-1+\beta)\lambda]+e^{t(-1+\beta)\lambda}\,P\,t^{1+n}\,\eta\text{trigger}\;\lambda\;\text{ExpIntegralE}[-n,\,t(-1+\beta)\lambda]\right)\right)$$

The total number of atoms in the excited state at any time t can be determined by adding $N_{1atom}(t)$ and $N_{2atom}(t)$. <u>The sum of $N_{1atom}(t)$ and $N_{2atom}(t)$ is</u>:

$$\frac{1}{\text{Etrigger}\,\beta\,\lambda}\left(P\,t^n\,\eta\text{trigger}+e^{t(-1+\beta)\lambda}\left(\text{Etrigger}\,\beta\,\lambda\,C[1]+P\,t^n\,\eta\text{trigger}\left(n\,\text{ExpIntegralE}[1-n,\,t(-1+\beta)\lambda]+t\,\lambda\,\text{ExpIntegralE}[-n,\,t(-1+\beta)\lambda]\right)\right)\right)$$

The total energy output at any time t can be determined by adding $E_1(t)$ and $E_2(t)$. <u>The sum of $E_1(t)$ and $E_2(t)$ is</u>:

$$\int_0^t \frac{1}{\text{Etrigger}^2\,\beta^2\,\lambda^2}\left(\eta\text{trigger}\left(P\,t^n\,\eta\text{trigger}+e^{t(-1+\beta)\lambda}\,\text{Etrigger}\,\beta\,\lambda\,C[1]+e^{t(-1+\beta)\lambda}\,n\,P\,t^n\,\eta\text{trigger}\;\text{ExpIntegralE}[1-n,\,t(-1+\beta)\lambda]+e^{t(-1+\beta)\lambda}\,P\,t^{1+n}\,\eta\text{trigger}\;\lambda\;\text{ExpIntegralE}[-n,\,t(-1+\beta)\lambda]\right)\right)dt$$

The total amount of stored energy, $E_{store}(t)$, can be determined from the product of the number of atoms in the excited state and the energy released per decay. <u>$E_{store}(t)$ is</u>:

$$\frac{1}{\text{Etrigger}\,\beta\,\lambda}\left(\text{Edecay}\,\eta\text{out}\left(P\,t^n\,\eta\text{trigger}+e^{t(-1+\beta)\lambda}\left(\text{Etrigger}\,\beta\,\lambda\,C[1]+P\,t^n\,\eta\text{trigger}\left(n\,\text{ExpIntegralE}[1-n,\,t(-1+\beta)\lambda]+t\,\lambda\,\text{ExpIntegralE}[-n,\,t(-1+\beta)\lambda]\right)\right)\right)\right)$$

Case 12 - Power = P (Sin[ω t + ϕ])^2

The value of local power consumption, $P_1(t)$, is:

P (Sin[ω t + ϕ])^2

The number of atoms, $N_{1atom}(t)$, required to produce the local power, can be determined from Equation (57). $N_{1atom}(t)$ is:

$$\frac{P\,\eta\text{trigger}\,\text{Sin}[\phi + t\,\omega]^2}{\text{Etrigger}\,\beta\,\lambda}$$

The cumulative energy, $E_1(t)$, produced for local power at time t can be determined by integrating the power output over time, Equation (58). $E_1(t)$ is:

$$\frac{P\,(2\,t\,\omega + \text{Sin}[2\,\phi] - \text{Sin}[2\,(\phi + t\,\omega)])}{4\,\omega}$$

The number of atoms required to supply the necessary feedback power, $N_{2atom}(t)$, can be determined by solving Equation (59). $N_{2atom}(t)$ is:

$$\frac{1}{2}\left(\frac{P\,\eta\text{trigger}}{-\text{Etrigger}\,\beta\,\lambda + \text{Etrigger}\,\beta^2\,\lambda} + 2\,e^{t\,(-1+\beta)\,\lambda}\,C[1] + \frac{P\,\eta\text{trigger}\,((-(-1+\beta)\,\lambda^2 + 4\,\omega^2)\,\text{Cos}[2\,(\phi + t\,\omega)] + 2\,\beta\,\lambda\,\omega\,\text{Sin}[2\,(\phi + t\,\omega)])}{\text{Etrigger}\,\beta\,\lambda\,((-1+\beta)^2\,\lambda^2 + 4\,\omega^2)}\right)$$

The power output required to supply the necessary feedback power, $P_2(t)$, can be determined from Equation (60). $P_2(t)$ is:

$$\frac{1}{2\,\text{Etrigger}\,\beta\,\lambda}\left(\eta\text{trigger}\left(\frac{P\,\eta\text{trigger}}{-\text{Etrigger}\,\beta\,\lambda + \text{Etrigger}\,\beta^2\,\lambda} + 2\,e^{t\,(-1+\beta)\,\lambda}\,C[1] + (P\,\eta\text{trigger}\,((-(-1+\beta)\,\lambda^2 + 4\,\omega^2)\,\text{Cos}[2\,(\phi + t\,\omega)] + 2\,\beta\,\lambda\,\omega\,\text{Sin}[2\,(\phi + t\,\omega)]))\,/\,(\text{Etrigger}\,\beta\,\lambda\,((-1+\beta)^2\,\lambda^2 + 4\,\omega^2))\right)\right)$$

The cumulative energy output required to supply the necessary feedback power, $E_2(t)$, can be determined from Equation (61). $E_2(t)$ is:

$$(\eta\text{trigger}\,(\omega\,((-1+\beta)^2\,\lambda^2 + 4\,\omega^2)\,(P\,t\,\eta\text{trigger} + 2\,(-1 + e^{t\,(-1+\beta)\,\lambda})\,\text{Etrigger}\,\beta\,C[1]) + P\,(-1+\beta)\,\eta\text{trigger}\,\text{Sin}[t\,\omega]\,((-(-1+\beta)\,\lambda^2 + 4\,\omega^2)\,\text{Cos}[2\,\phi + t\,\omega] + 2\,\beta\,\lambda\,\omega\,\text{Sin}[2\,\phi + t\,\omega])))\,/\,(2\,\text{Etrigger}^2\,\beta^2\,\lambda^2\,((-1+\beta)^3\,\lambda^2\,\omega + 4\,(-1+\beta)\,\omega^3))$$

The combined power output from $P_1(t)$ and $P_2(t)$ can be determined by adding the two terms. The sum of $P_1(t)$ and $P_2(t)$ is:

$$\frac{1}{\text{Etrigger }\beta\lambda}\left(\eta\text{trigger}\left(\frac{\text{P }\eta\text{trigger Sin}[\phi+t\,\omega]^2}{\text{Etrigger }\beta\lambda}+\frac{1}{2}\left(\frac{\text{P }\eta\text{trigger}}{-\text{Etrigger }\beta\lambda+\text{Etrigger }\beta^2\lambda}+2\,e^{t\,(-1+\beta)\,\lambda}\,C[1]+(\text{P }\eta\text{trigger }((-(-1+\beta)\,\lambda^2+4\,\omega^2)\,\text{Cos}[2\,(\phi+t\,\omega)]+2\,\beta\,\lambda\,\omega\,\text{Sin}[2\,(\phi+t\,\omega)]))\,/\,(\text{Etrigger }\beta\lambda\,((-1+\beta)^2\,\lambda^2+4\,\omega^2))\right)\right)\right)$$

The total number of atoms in the excited state at any time t can be determined by adding $N_{1atom}(t)$ and $N_{2atom}(t)$. <u>The sum of $N_{1atom}(t)$ and $N_{2atom}(t)$ is</u>:

$$\frac{\text{P }\eta\text{trigger Sin}[\phi+t\,\omega]^2}{\text{Etrigger }\beta\lambda}+\frac{1}{2}\left(\frac{\text{P }\eta\text{trigger}}{-\text{Etrigger }\beta\lambda+\text{Etrigger }\beta^2\lambda}+2\,e^{t\,(-1+\beta)\,\lambda}\,C[1]+\frac{\text{P }\eta\text{trigger }((-(-1+\beta)\,\lambda^2+4\,\omega^2)\,\text{Cos}[2\,(\phi+t\,\omega)]+2\,\beta\,\lambda\,\omega\,\text{Sin}[2\,(\phi+t\,\omega)])}{\text{Etrigger }\beta\lambda\,((-1+\beta)^2\,\lambda^2+4\,\omega^2)}\right)$$

The total energy output at any time t can be determined by adding $E_1(t)$ and $E_2(t)$. <u>The sum of $E_1(t)$ and $E_2(t)$ is</u>:

$$\frac{1}{2\,\text{Etrigger}^2\,\beta\,\lambda^2}\left(\eta\text{trigger}\left(\frac{\text{P t }\eta\text{trigger}+2\,(-1+e^{t\,(-1+\beta)\,\lambda})\,\text{Etrigger }C[1]}{-1+\beta}+\frac{\text{P }\eta\text{trigger }\lambda\,\text{Sin}[t\,\omega]\,(-(-1+\beta)\,\lambda\,\text{Cos}[2\,\phi+t\,\omega]+2\,\omega\,\text{Sin}[2\,\phi+t\,\omega])}{(-1+\beta)^2\,\lambda^2\,\omega+4\,\omega^3}\right)\right)$$

The total amount of stored energy, $E_{store}(t)$, can be determined from the product of the number of atoms in the excited state and the energy released per decay. <u>$E_{store}(t)$ is</u>:

$$\text{Edecay }\eta\text{out}\left(\frac{\text{P }\eta\text{trigger Sin}[\phi+t\,\omega]^2}{\text{Etrigger }\beta\lambda}+\frac{1}{2}\left(\frac{\text{P }\eta\text{trigger}}{-\text{Etrigger }\beta\lambda+\text{Etrigger }\beta^2\lambda}+2\,e^{t\,(-1+\beta)\,\lambda}\,C[1]+(\text{P }\eta\text{trigger }((-(-1+\beta)\,\lambda^2+4\,\omega^2)\,\text{Cos}[2\,(\phi+t\,\omega)]+2\,\beta\,\lambda\,\omega\,\text{Sin}[2\,(\phi+t\,\omega)]))\,/\,(\text{Etrigger }\beta\lambda\,((-1+\beta)^2\,\lambda^2+4\,\omega^2))\right)\right)$$

Case 13 - Power = P Erfc[t]

The value of local power consumption, P_1(t), is:

P Erfc[x]

The number of atoms, N_{1atom}(t), required to produce the local power, can be determined from Equation (57). N_{1atom}(t) is:

$$\frac{P\,\eta\text{trigger Erfc}[t]}{\text{Etrigger}\,\beta\,\lambda}$$

The cumulative energy, E_1(t), produced for local power at time t can be determined by integrating the power output over time, Equation (58). E_1(t) is:

$$P\left(\frac{1-e^{-t^2}}{\sqrt{\pi}}+t-t\,\text{Erf}[t]\right)$$

The number of atoms required to supply the necessary feedback power, N_{2atom}(t), can be determined by solving Equation (59). N_{2atom}(t) is:

$$\frac{1}{\text{Etrigger}\,(-1+\beta)\,\beta\,\lambda}\left(e^{t\,(-1+\beta)\,\lambda}\,\text{Etrigger}\,(-1+\beta)\,\beta\,\lambda\,C[1]+P\,\eta\text{trigger}\left(e^{\frac{1}{4}\,(-1+\beta)\,\lambda\,(4\,t+(-1+\beta)\,\lambda)}\,\beta\,\text{Erf}\left[t+\frac{1}{2}\,(-1+\beta)\,\lambda\right]+\text{Erfc}[t]\right)\right)$$

The power output required to supply the necessary feedback power, P_2(t), can be determined from Equation (60). P_2(t) is:

$$\frac{1}{\text{Etrigger}^2\,(-1+\beta)\,\beta^2\,\lambda^2}\left(\eta\text{trigger}\left(e^{t\,(-1+\beta)\,\lambda}\,\text{Etrigger}\,(-1+\beta)\,\beta\,\lambda\,C[1]+P\,\eta\text{trigger}\left(e^{\frac{1}{4}\,(-1+\beta)\,\lambda\,(4\,t+(-1+\beta)\,\lambda)}\,\beta\,\text{Erf}\left[t+\frac{1}{2}\,(-1+\beta)\,\lambda\right]+\text{Erfc}[t]\right)\right)\right)$$

The cumulative energy output required to supply the necessary feedback power, E_2(t), can be determined from Equation (61). E_2(t) is:

$$\frac{1}{\text{Etrigger}^2\,(-1+\beta)^2\,\lambda^3}\Bigg(\eta\text{trigger}\Bigg((-1+e^{t\,(-1+\beta)\,\lambda})\,\text{Etrigger}\,\lambda\,C[1]-\frac{(-1+e^{t\,(-1+\beta)\,\lambda})\,\text{Etrigger}\,\lambda\,C[1]}{\beta}+\frac{P\,(-1+\beta)\,\eta\text{trigger}\,\lambda\left(\frac{1-e^{-t^2}}{\sqrt{\pi}}+t-t\,\text{Erf}[t]\right)}{\beta^2}-\frac{1}{\beta}\left(P\,\eta\text{trigger}\left(\text{Erf}[t]+e^{\frac{1}{4}(-1+\beta)^2\lambda^2}\,\text{Erf}\left[\frac{1}{2}(-1+\beta)\,\lambda\right]-e^{\frac{1}{4}(-1+\beta)\,\lambda\,(4\,t+(-1+\beta)\,\lambda)}\,\text{Erf}\left[t+\frac{1}{2}(-1+\beta)\,\lambda\right]\right)\right)\Bigg)\Bigg)$$

The combined power output from $P_1(t)$ and $P_2(t)$ can be determined by adding the two terms. <u>The sum of $P_1(t)$ and $P_2(t)$ is</u>:

$$\frac{1}{\text{Etrigger}^2\,(-1+\beta)\,\beta\,\lambda^2}\left(\eta\text{trigger}\left(e^{t\,(-1+\beta)\,\lambda}\,\text{Etrigger}\,(-1+\beta)\,\lambda\,C[1]+e^{\frac{1}{4}(-1+\beta)\,\lambda\,(4\,t+(-1+\beta)\,\lambda)}\,P\,\eta\text{trigger}\,\text{Erf}\left[t+\frac{1}{2}(-1+\beta)\,\lambda\right]+P\,\eta\text{trigger}\,\text{Erfc}[t]\right)\right)$$

The total number of atoms in the excited state at any time t can be determined by adding $N_{1atom}(t)$ and $N_{2atom}(t)$. <u>The sum of $N_{1atom}(t)$ and $N_{2atom}(t)$ is</u>:

$$\frac{1}{\text{Etrigger}^2\,(-1+\beta)^2\,\lambda^3}\Bigg(\eta\text{trigger}\Bigg((-1+e^{t\,(-1+\beta)\,\lambda})\,\text{Etrigger}\,\lambda\,C[1]-\frac{(-1+e^{t\,(-1+\beta)\,\lambda})\,\text{Etrigger}\,\lambda\,C[1]}{\beta}+\frac{P\,(-1+\beta)\,\eta\text{trigger}\,\lambda\left(\frac{1-e^{-t^2}}{\sqrt{\pi}}+t-t\,\text{Erf}[t]\right)}{\beta}-\frac{1}{\beta}\left(P\,\eta\text{trigger}\left(\text{Erf}[t]+e^{\frac{1}{4}(-1+\beta)^2\lambda^2}\,\text{Erf}\left[\frac{1}{2}(-1+\beta)\,\lambda\right]-e^{\frac{1}{4}(-1+\beta)\,\lambda\,(4\,t+(-1+\beta)\,\lambda)}\,\text{Erf}\left[t+\frac{1}{2}(-1+\beta)\,\lambda\right]\right)\right)\Bigg)\Bigg)$$

The total amount of stored energy, $E_{store}(t)$, can be determined from the product of the number of atoms in the excited state and the energy released per decay. <u>$E_{store}(t)$ is</u>:

$$\frac{1}{\text{Etrigger }(-1+\beta)\,\lambda}\left(\text{Edecay }\eta\text{out }\left(e^{t\,(-1+\beta)\,\lambda}\,\text{Etrigger }(-1+\beta)\,\lambda\,C[1]+e^{\frac{1}{4}\,(-1+\beta)\,\lambda\,(4\,t+(-1+\beta)\,\lambda)}\right.\right.$$

$$\left.\left.P\,\eta\text{trigger Erf}\left[t+\frac{1}{2}\,(-1+\beta)\,\lambda\right]+P\,\eta\text{trigger Erfc}[t]\right)\right)$$

Case 14 - Power = P Erf[t]

The value of local power consumption, $P_1(t)$, is:

P Erf[x]

The number of atoms, $N_{1atom}(t)$, required to produce the local power, can be determined from Equation (57). $N_{1atom}(t)$ is:

$$\frac{P\,\eta\text{trigger Erf}[t]}{\text{Etrigger }\beta\,\lambda}$$

The cumulative energy, $E_1(t)$, produced for local power at time t can be determined by integrating the power output over time, Equation (58). $E_1(t)$ is:

$$P\left(\frac{-1+e^{-t^2}}{\sqrt{\pi}}+t\,\text{Erf}[t]\right)$$

The number of atoms required to supply the necessary feedback power, $N_{2atom}(t)$, can be determined by solving Equation (59). $N_{2atom}(t)$ is:

$$\frac{1}{\text{Etrigger }(-1+\beta)\,\beta\,\lambda}\left(P\,\eta\text{trigger Erf}[t]+\beta\left(e^{t\,(-1+\beta)\,\lambda}\,\text{Etrigger }(-1+\beta)\,\lambda\,C[1]-e^{\frac{1}{4}(-1+\beta)\,\lambda\,(4\,t+(-1+\beta)\,\lambda)}\,P\,\eta\text{trigger Erf}\left[t+\frac{1}{2}(-1+\beta)\,\lambda\right]\right)\right)$$

The power output required to supply the necessary feedback power, $P_2(t)$, can be determined from Equation (60). $P_2(t)$ is:

$$\frac{1}{\text{Etrigger}^2\,(-1+\beta)\,\beta^2\,\lambda^2}\left(\eta\text{trigger}\left(P\,\eta\text{trigger Erf}[t]+\beta\left(e^{t\,(-1+\beta)\,\lambda}\,\text{Etrigger }(-1+\beta)\,\lambda\,C[1]-e^{\frac{1}{4}(-1+\beta)\,\lambda\,(4\,t+(-1+\beta)\,\lambda)}\,P\,\eta\text{trigger Erf}\left[t+\frac{1}{2}(-1+\beta)\,\lambda\right]\right)\right)\right)$$

The cumulative energy output required to supply the necessary feedback power, $E_2(t)$, can be determined from Equation (61). $E_2(t)$ is:

$$\frac{1}{\text{Etrigger}^2\,(-1+\beta)^2\,\lambda^3}\Bigg(\eta\text{trigger}\Bigg(\Big((-1+e^{t(-1+\beta)\lambda})\,\text{Etrigger}\,\lambda\,C[1]-\frac{(-1+e^{t(-1+\beta)\lambda})\,\text{Etrigger}\,\lambda\,C[1]}{\beta}+$$

$$\frac{P\,(-1+\beta)\,\eta\text{trigger}\,\lambda\left(\frac{-1+e^{-t^2}}{\sqrt{\pi}}+t\,\text{Erf}[t]\right)}{\beta^2}+$$

$$\frac{1}{\beta}\Big(P\,\eta\text{trigger}\,\Big(\text{Erf}[t]+e^{\frac{1}{4}(-1+\beta)^2\lambda^2}\,\text{Erf}\Big[\frac{1}{2}(-1+\beta)\,\lambda\Big]-$$

$$e^{\frac{1}{4}(-1+\beta)\lambda(4t+(-1+\beta)\lambda)}\,\text{Erf}\Big[t+\frac{1}{2}(-1+\beta)\,\lambda\Big]\Big)\Big)\Bigg)\Bigg)$$

The combined power output from $P_1(t)$ and $P_2(t)$ can be determined by adding the two terms. <u>The sum of $P_1(t)$ and $P_2(t)$ is</u>:

$$\frac{1}{\text{Etrigger}^2\,(-1+\beta)\,\beta\,\lambda^2}$$

$$\Big(\eta\text{trigger}\,\Big(e^{t(-1+\beta)\lambda}\,\text{Etrigger}\,(-1+\beta)\,\lambda\,C[1]+P\,\eta\text{trigger}\,\text{Erf}[t]-$$

$$e^{\frac{1}{4}(-1+\beta)\lambda(4t+(-1+\beta)\lambda)}\,P\,\eta\text{trigger}\,\text{Erf}\Big[t+\frac{1}{2}(-1+\beta)\,\lambda\Big]\Big)\Big)$$

The total number of atoms in the excited state at any time t can be determined by adding $N_{1atom}(t)$ and $N_{2atom}(t)$. <u>The sum of $N_{1atom}(t)$ and $N_{2atom}(t)$ is</u>:

$$\frac{1}{\text{Etrigger}\,(-1+\beta)\,\lambda}\Big(e^{t(-1+\beta)\lambda}\,\text{Etrigger}\,(-1+\beta)\,\lambda\,C[1]+$$

$$P\,\eta\text{trigger}\,\text{Erf}[t]-e^{\frac{1}{4}(-1+\beta)\lambda(4t+(-1+\beta)\lambda)}\,P\,\eta\text{trigger}\,\text{Erf}\Big[t+\frac{1}{2}(-1+\beta)\,\lambda\Big]\Big)$$

The total energy output at any time t can be determined by adding $E_1(t)$ and $E_2(t)$. <u>The sum of $E_1(t)$ and $E_2(t)$ is</u>:

$$\frac{1}{\text{Etrigger}^2\,(-1+\beta)^2\,\beta\,\lambda^3}\Big(\eta\text{trigger}\,\Big(\frac{1}{\sqrt{\pi}}\Big(e^{-t^2}\,(-1+\beta)\,\lambda$$

$$\Big(P\,\eta\text{trigger}+e^{t^2}\,\Big(-P\,\eta\text{trigger}+(-1+e^{t(-1+\beta)\lambda})\,\text{Etrigger}\,\sqrt{\pi}\,C[1]\Big)\Big)\Big)+$$

$$P\,\eta\text{trigger}\,\Big((1+t\,(-1+\beta)\,\lambda)\,\text{Erf}[t]+e^{\frac{1}{4}(-1+\beta)^2\lambda^2}\,\text{Erf}\Big[\frac{1}{2}(-1+\beta)\,\lambda\Big]-$$

$$e^{\frac{1}{4}(-1+\beta)\lambda(4t+(-1+\beta)\lambda)}\,\text{Erf}\Big[t+\frac{1}{2}(-1+\beta)\,\lambda\Big]\Big)\Big)\Big)$$

The total amount of stored energy, $E_{store}(t)$, can be determined from the product of the number of atoms in the excited state and the energy released per decay. <u>$E_{store}(t)$ is</u>:

$$\frac{1}{\text{Etrigger}\ (-1+\beta)\ \lambda}$$
$$\left(\text{Edecay}\ \eta\text{out}\ \left(e^{t\ (-1+\beta)\ \lambda}\ \text{Etrigger}\ (-1+\beta)\ \lambda\ C[1] + P\ \eta\text{trigger}\ \text{Erf}[t] - e^{\frac{1}{4}\ (-1+\beta)\ \lambda\ (4\ t+(-1+\beta)\ \lambda)}\ P\ \eta\text{trigger}\ \text{Erf}\left[t+\frac{1}{2}\ (-1+\beta)\ \lambda\right]\right)\right)$$

Appendix B

Values of the function $G(t)$ and $[G(t, t_b) - G(t, t_a)]$
for use in evaluating periodic power outputs

Case 1 - Constant Power, P

G(t, tau) is:

$$\frac{e^{-(t-tau)\ (-1+\beta)\ \lambda}\ P}{(-1+\beta)\ \lambda}$$

G(t, ta, tb) is:

$$\frac{e^{-t\ (-1+\beta)\ \lambda}\ (-e^{ta\ (-1+\beta)\ \lambda} + e^{tb\ (-1+\beta)\ \lambda})\ P}{(-1+\beta)\ \lambda}$$

Case 2 - Power increases linearly with time, P t

G(t, tau) is:

$$\frac{e^{-(t-tau)\ (-1+\beta)\ \lambda}\ P\ (-1 + tau\ (-1+\beta)\ \lambda)}{(-1+\beta)^2\ \lambda^2}$$

G(t, ta, tb) is:

$$-\frac{1}{(-1+\beta)^2\ \lambda^2}\ (e^{-t\ (-1+\beta)\ \lambda}\ P\ (e^{ta\ (-1+\beta)\ \lambda}\ (-1 + ta\ (-1+\beta)\ \lambda) + e^{tb\ (-1+\beta)\ \lambda}\ (1 + tb\ (\lambda - \beta\ \lambda))))$$

Case 3 - Power increases with the square of time, P t^2

G(t, tau) is:

$$\frac{e^{-(t-tau)\ (-1+\beta)\ \lambda}\ P\ (2 - 2\ tau\ (-1+\beta)\ \lambda + tau^2\ (-1+\beta)^2\ \lambda^2)}{(-1+\beta)^3\ \lambda^3}$$

G(t, ta, tb) is:

$$-\frac{1}{(-1+\beta)^3\ \lambda^3}\ (e^{-t\ (-1+\beta)\ \lambda}\ P\ (e^{ta\ (-1+\beta)\ \lambda}\ (2 - 2\ ta\ (-1+\beta)\ \lambda + ta^2\ (-1+\beta)^2\ \lambda^2) - e^{tb\ (-1+\beta)\ \lambda}\ (2 - 2\ tb\ (-1+\beta)\ \lambda + tb^2\ (-1+\beta)^2\ \lambda^2)))$$

Case 4 - Power with the cube of time, P t^3

G(t, tau) is:

$$\frac{1}{(-1+\beta)^4 \lambda^4} \left(e^{-(t-tau)\,(-1+\beta)\,\lambda}\, P \left(-6 + 6\, tau\, (-1+\beta)\,\lambda - 3\, tau^2\, (-1+\beta)^2 \lambda^2 + tau^3\, (-1+\beta)^3 \lambda^3\right)\right)$$

G(t, ta, tb) is:

$$-\frac{1}{(-1+\beta)^4 \lambda^4} \left(e^{-t\,(-1+\beta)\,\lambda}\, P \left(e^{ta\,(-1+\beta)\,\lambda} \left(-6 + 6\, ta\, (-1+\beta)\,\lambda - 3\, ta^2\, (-1+\beta)^2 \lambda^2 + ta^3\, (-1+\beta)^3 \lambda^3\right) + e^{tb\,(-1+\beta)\,\lambda} \left(6 - 6\, tb\, (-1+\beta)\,\lambda + 3\, tb^2\, (-1+\beta)^2 \lambda^2 - tb^3\, (-1+\beta)^3 \lambda^3\right)\right)\right)$$

Case 5 - Power varies as P Sin(ω t + ϕ)

G(t, tau) is:

$$\frac{e^{-(t-tau)\,(-1+\beta)\,\lambda}\, P\, \mathrm{Sin}[\phi + t\,\omega]}{(-1+\beta)\,\lambda}$$

G(t, ta, tb) is:

$$-\frac{e^{-t\,(-1+\beta)\,\lambda} \left(e^{ta\,(-1+\beta)\,\lambda} - e^{tb\,(-1+\beta)\,\lambda}\right) P\, \mathrm{Sin}[\phi + t\,\omega]}{(-1+\beta)\,\lambda}$$

Case 6 - Power varies as P Exp[θ t]

G(t, tau) is:

$$\frac{e^{tau\,(-1+\beta)\,\lambda + t\,(\theta+\lambda-\beta\,\lambda)}\, P}{(-1+\beta)\,\lambda}$$

G(t, ta, tb) is:

$$-\frac{e^{t\,(\theta+\lambda-\beta\,\lambda)} \left(e^{ta\,(-1+\beta)\,\lambda} - e^{tb\,(-1+\beta)\,\lambda}\right) P}{(-1+\beta)\,\lambda}$$

Case 7 - Power varies as P t^n

G(t, tau) is:

$$-e^{-t(-1+\beta)\lambda}\,P\,\mathrm{tau}^{1+n}\,(-\mathrm{tau}\,(-1+\beta)\,\lambda)^{-1-n}\,\mathrm{Gamma}[1+n,\,-\mathrm{tau}\,(-1+\beta)\,\lambda]$$

G(t, ta, tb) is:

$$-e^{-t(-1+\beta)\lambda}\,P\,\mathrm{ta}\,(-\mathrm{ta}\,(-1+\beta)\,\lambda)^{-1-n}\,(-\mathrm{tb}\,(-1+\beta)\,\lambda)^{-n}\,(-\mathrm{ta}^{n}\,(-\mathrm{tb}\,(-1+\beta)\,\lambda)^{n}\,\mathrm{Gamma}[1+n,\,-\mathrm{ta}\,(-1+\beta)\,\lambda]+\mathrm{tb}^{n}\,(-\mathrm{ta}\,(-1+\beta)\,\lambda)^{n}\,\mathrm{Gamma}[1+n,\,-\mathrm{tb}\,(-1+\beta)\,\lambda])$$

Case 8 - Power varies as P Cos(ω t + ϕ)

G(t, tau) is:

$$\frac{e^{-(t-\mathrm{tau})(-1+\beta)\lambda}\,P\,((-1+\beta)\,\lambda\,\mathrm{Cos}[\phi+\mathrm{tau}\,\omega]+\omega\,\mathrm{Sin}[\phi+\mathrm{tau}\,\omega])}{(-1+\beta)^{2}\,\lambda^{2}+\omega^{2}}$$

G(t, ta, tb) is:

$$-\frac{1}{(-1+\beta)^{2}\,\lambda^{2}+\omega^{2}}\,(e^{-t(-1+\beta)\lambda}\,P\,(e^{\mathrm{ta}(-1+\beta)\lambda}\,(-1+\beta)\,\lambda\,\mathrm{Cos}[\phi+\mathrm{ta}\,\omega]-e^{\mathrm{tb}(-1+\beta)\lambda}\,(-1+\beta)\,\lambda\,\mathrm{Cos}[\phi+\mathrm{tb}\,\omega]+\omega\,(e^{\mathrm{ta}(-1+\beta)\lambda}\,\mathrm{Sin}[\phi+\mathrm{ta}\,\omega]-e^{\mathrm{tb}(-1+\beta)\lambda}\,\mathrm{Sin}[\phi+\mathrm{tb}\,\omega])))$$

Case 9 - Power varies as P / t

G(t, tau) is:

$$e^{-t(-1+\beta)\lambda}\,P\,\mathrm{ExpIntegralEi}[\mathrm{tau}\,(-1+\beta)\,\lambda]$$

G(t, ta, tb) is:

$$e^{-t(-1+\beta)\lambda}\,P\,(-\mathrm{ExpIntegralEi}[\mathrm{ta}\,(-1+\beta)\,\lambda]+\mathrm{ExpIntegralEi}[\mathrm{tb}\,(-1+\beta)\,\lambda])$$

Case 10 - Power varies as P / t^2

G(t, tau) is:

$$-\frac{1}{\text{tau}}\left(e^{-t\,(-1+\beta)\,\lambda}\,P\left(e^{\text{tau}\,(-1+\beta)\,\lambda}-\text{tau}\,(-1+\beta)\,\lambda\,\text{ExpIntegralEi}[\text{tau}\,(-1+\beta)\,\lambda]\right)\right)$$

G(t, ta, tb) is:

$$\frac{1}{\text{ta tb}}\left(e^{-t\,(-1+\beta)\,\lambda}\,P\left(-e^{\text{tb}\,(-1+\beta)\,\lambda}\,\text{ta}+e^{\text{ta}\,(-1+\beta)\,\lambda}\,\text{tb}-\text{ta tb}\,(-1+\beta)\,\lambda\,\text{ExpIntegralEi}[\text{ta}\,(-1+\beta)\,\lambda]+\text{ta tb}\,(-1+\beta)\,\lambda\,\text{ExpIntegralEi}[\text{tb}\,(-1+\beta)\,\lambda]\right)\right)$$

Case 11 - Power varies as P Sin(ω t + ϕ) +K

G(t, tau) is:

$$K\,\text{tau}+\frac{e^{-(t-\text{tau})\,(-1+\beta)\,\lambda}\,P\,\text{Sin}[\phi+t\,\omega]}{(-1+\beta)\,\lambda}$$

G(t, ta, tb) is:

$$\frac{1}{(-1+\beta)\,\lambda}\left(e^{-t\,(-1+\beta)\,\lambda}\left(-e^{t\,(-1+\beta)\,\lambda}\,K\,(\text{ta}-\text{tb})\,(-1+\beta)\,\lambda+\left(-e^{\text{ta}\,(-1+\beta)\,\lambda}+e^{\text{tb}\,(-1+\beta)\,\lambda}\right)P\,\text{Sin}[\phi+t\,\omega]\right)\right)$$

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